

ELLIPTIC FIBRATIONS ON A GENERIC JACOBIAN KUMMER SURFACE

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ABSTRACT. We describe all the elliptic fibrations with section on the Kummer surface X of the Jacobian of a very general curve C of genus 2 over an algebraically closed field of characteristic zero, modulo the automorphism group of X and the symmetric group on the Weierstrass points of C . In particular, we compute elliptic parameters and Weierstrass equations for the twenty five different fibrations, and analyze the reducible fibers and Mordell-Weil lattices. This answers completely a question of Kuwata and Shioda [KS].

1. INTRODUCTION

In this paper we will analyze elliptic fibrations on a specific class of K3 surfaces. In particular, for us a K3 surface X will be a projective algebraic nonsingular surface over a field k of characteristic 0, for which $H^1(X, \mathcal{O}_X) = 0$ and the canonical bundle $K_X \cong \mathcal{O}_X$. These are one of the types of surfaces of Kodaira dimension 0. Another important class which appears in the classification of surfaces is that of elliptic surfaces: those surfaces for which there is a fibration $X \rightarrow \mathbb{P}^1$ of curves of arithmetic genus 1.

Not all K3 surfaces are elliptic: K3 surfaces equipped with a polarization of a fixed degree $2n$ are described by a 19-dimensional moduli space, and those admitting an elliptic fibration correspond to points of a countable union of subvarieties of codimension 1 in this moduli space.

On the other hand, one may associate a discrete parameter $n = \chi(\mathcal{O}_E) = \frac{1}{12}\chi_{top}(E)$ to an elliptic surface E . The case $n = 1$ corresponds to rational elliptic surfaces, $n = 2$ to elliptic K3 surfaces, and $n \geq 3$ to “honestly” elliptic surfaces, i.e. those of Kodaira dimension 1.

It is relatively easy, in principle, to describe elliptic fibrations on a K3 surface X , as follows. The notions of linear, algebraic and numerical equivalence for divisors are equivalent for a K3 surface, and the Néron-Severi group $NS(X)$ which describes the algebraic divisors modulo this equivalence relation is torsion-free and a lattice under the intersection pairing. It is an even lattice, and has signature $(1, r - 1)$ for some $1 \leq r \leq 20$, by the Hodge index theorem. A classical theorem of Piatetski-Shapiro and Shafarevich [PSS, pp. 559–560] establishes a correspondence between elliptic fibrations on X and primitive nef elements of $NS(X)$ of self-intersection 0. We choose the term elliptic divisor (class) to describe an effective divisor (class) of self-intersection zero.

Another geometrical property of a K3 surface over \mathbb{C} which is explicitly computable, in principle, is its automorphism group. Here the Torelli theorem of Piatetski-Shapiro and Shafarevich [PSS] for algebraic K3 surfaces stipulates that an element of $\text{Aut}(X)$ is completely specified by its action on $H^2(X, \mathbb{Z})$. Conversely, an automorphism of the lattice $H^2(X, \mathbb{Z})$ which respects the Hodge decomposition and preserves the ample or Kähler cone (alternatively, preserves the classes of effective divisors) gives rise to an automorphism of X .

A theorem of Sterk [St] guarantees that for a K3 surface X over \mathbb{C} , and for any even integer $d \geq -2$, there are only finitely many divisor classes of self-intersection d modulo $\text{Aut}(X)$. In particular, there are only finitely many elliptic fibrations modulo the automorphism group. In practice, however, it may be quite difficult to compute all the elliptic fibrations for any given K3 surface (for example, as a double cover of \mathbb{P}^2 , given by $y^2 = f(x_0, x_1, x_2)$, or as a quartic surface in \mathbb{P}^3 , given by $f(x_0, x_1, x_2, x_3) = 0$). This is because it might not be easy to compute the Néron-Severi group, or to understand $NS(X) \subset H^2(X, \mathbb{Z})$ well enough to compute the automorphism group, or even knowing these two, to compute all the distinct orbits of $\text{Aut}(X)$ on the elliptic divisors.

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In this paper, we carry out this computation for a case of geometric and arithmetic interest: that of a (desingularized) Kummer surface of a Jacobian of a genus 2 curve C whose Jacobian $J(C)$ has no extra endomorphisms. In this paper, we will reserve the term “generic Jacobian Kummer surface” and the label $\text{Km}(J(C))$ for such a Kummer surface. We also focus on elliptic fibrations with a section (also called Jacobian elliptic fibrations, though we will not use this term since “Jacobian” is being used in a different context), although the calculations in the first part of the paper in principle will also allow us to describe the genus 1 fibrations without a section. Our main theorem states that there are essentially twenty-five different elliptic fibrations with section on a generic Jacobian Kummer surface. Furthermore, we compute elliptic parameters and Weierstrass equations for each of these elliptic fibrations, describe the reducible fibers, and give a basis of the sections generating the Mordell-Weil lattice in each case. Apart from solving a classification problem, we expect that these results will facilitate further calculations of Kummer surfaces and associated K3 surfaces and their moduli spaces. Another application is in the construction of elliptic surfaces and curves of high rank. For instance, the very last elliptic fibration described in this paper gives a family of elliptic K3 surfaces of Mordell-Weil rank 6 over the moduli space of genus 2 curves with full 2-level structure. In fact, this family descends to the moduli space \mathcal{M}_2 of genus 2 curves without 2-level structure, giving a family of elliptic K3 surfaces of geometric Mordell-Weil rank 6. Finally, many of the equations described in this paper give new models of Kummer surfaces as double plane sextics. Note that though these calculations are carried out for a very general curve of genus 2, in most cases the same or analogous calculation may be carried out for special genus 2 curves, resulting in the same Weierstrass equations. The presence of extra endomorphisms will manifest itself in an altered configuration of reducible fibers, or by a jump in the Mordell-Weil rank. Finally, we note that although we work in characteristic zero, the classification result should also hold (with minor modification to the formulas given here) in odd positive characteristic: since the K3 surfaces considered are non-supersingular (by the genericity assumption), they lift to characteristic zero by Deligne’s lifting result [D].

We now mention some related results. Recall that the two types of principally polarized abelian surfaces are Jacobians of genus 2, and products of two elliptic curves. A natural question is to compute all the elliptic fibrations on the Kummer surfaces of such abelian surfaces, subject to genericity assumptions. Oguiso [O] classified the different elliptic fibrations (modulo the automorphism group) on $\text{Km}(E_1 \times E_2)$, when E_1 and E_2 are non-isogenous elliptic curves. Nishiyama [N] obtained the result by a more algebraic method, and also extended it to the Kummer surface of the product of an elliptic curve with itself. Keum and Kondo [KK] computed the automorphism group of such a Kummer surface. Kuwata and Shioda [KS] have more recently computed elliptic parameters and Weierstrass equations for $\text{Km}(E_1 \times E_2)$. They asked the question which motivated this paper.

Keum [Ke1] found an elliptic fibration on $\text{Km}(J(C))$ with $D_6A_3A_1^6$ reducible fibers, and Mordell-Weil group $(\mathbb{Z}/2\mathbb{Z})^2$. Shioda [Sh1] found an elliptic fibration on $\text{Km}(J(C))$ with $D_4^2A_1^6$ reducible fibers, and Mordell-Weil group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. In previous work [Ku], the author found a further elliptic fibration with $D_9A_1^6$ reducible fibers and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$. The automorphism group of $\text{Km}(J(C))$ was computed by Kondo [Ko], building on work of Keum [Ke2].

2. KUMMER SURFACE

Let C be a curve of genus 2 over an algebraically closed field k of characteristic different from 2. We review some of the classical geometry of the surface $\text{Km}(J(C))$ (see [Hud] for more background).

The linear system divisor 2Θ , where Θ is an embedding of C in $J(C)$ defining the principal polarization, maps the Jacobian of C to \mathbb{P}^3 by a map of degree 2. The image is a quartic surface with 16 singularities, the singular model of the Kummer surface of $J(C)$. The 16 singular points are ordinary double points on the quartic surface, and are called **nodes**. The node n_0 comes from the identity or 0 point of the Jacobian, whereas the node n_{ij} comes from the 2-torsion point which is the difference of divisors $[(\theta_i, 0)] - [(\theta_j, 0)]$ corresponding to two distinct Weierstrass points on C . The nodes n_{0i} are usually abbreviated n_i .

There are also sixteen hyperplanes in \mathbb{P}^3 which are tangent to the Kummer quartic. These are called **tropes**. Each trope intersects the quartic in a conic with multiplicity 2 (this conic is also called a trope, by abuse of notation), and contains 6 nodes. Conversely, each node is contained in exactly 6 tropes. This beautiful configuration is called the (16, 6) Kummer configuration.

The Néron-Severi lattice of the nonsingular Kummer contains classes of rational curves N_i and N_{ij} which are -2 -curves obtained by blowing up the nodes of the singular Kummer, and also the classes of the T_i and T_{ij} coming from the tropes. Henceforth we will use the terminology N_μ to denote the class of a node, and similarly T_μ for the class of a trope. We will denote the lattice generated by these as $\Lambda_{(16,6)}$ (it is abstractly isomorphic to $\langle 4 \rangle \oplus D_8(-1)^2$ [Ke2]). It has signature $(1, 16)$ and discriminant 2^6 and is always contained primitively in the Néron-Severi lattice of a Kummer surface of a curve of genus 2. For a very general point of the moduli space of genus 2 curves or of principally polarized abelian surface (i.e. away from a countable union of hypersurfaces), the Néron-Severi lattice of the Kummer surface is actually isomorphic to $\Lambda_{(16,6)}$.

Let H be the class of a nonsingular hyperplane section of the singular Kummer surface in \mathbb{P}^3 . We have the following relations in the Néron-Severi lattice of the Kummer surface X :

$$\begin{aligned} T_0 &= \frac{1}{2} \left(H - \sum_{i=0}^5 N_i \right), & T_i &= \frac{1}{2} \left(H - N_i - \sum_{j \neq i} N_{ij} \right) \\ T_{ij} &= \frac{1}{2} (H - N_i - N_j - N_{ij} - N_{lm} - N_{mn} - N_{ln}) \end{aligned}$$

where $\{l, m, n\}$ is the complementary set to $\{0, i, j\}$ in $\{0, 1, 2, 3, 4, 5\}$.

From these and the inner products

$$\begin{aligned} H^2 &= 4 & N_\mu^2 &= -2 \\ H \cdot N_\mu &= 0 & N_\mu \cdot N_\nu &= 0 \text{ for } \mu \neq \nu \end{aligned}$$

we can easily compute all the inner products between the classes of the nodes and the tropes.

3. ELLIPTIC FIBRATIONS ON A GENERIC KUMMER SURFACE

3.1. Automorphism group. We recall here some facts about the automorphism group of the generic Jacobian Kummer surface. A few of these are classical [Hud, Hut, Kl], though the setup we will need is much more recent, coming from work of Keum [Ke2] and Kondo [Ko].

The classical automorphisms are the following:

- (1) Sixteen translations: The translations by 2-torsion points on the abelian surface give rise to involutions on the Kummer surface. These are linear i.e. induced by elements of $PGL_4(k)$ on the singular Kummer surface in \mathbb{P}^3 .
- (2) Sixteen projections: Projecting the singular quartic Kummer surface Y from any of the nodes n_α gives rise to a double cover $Y \rightarrow \mathbb{P}^2$. The interchange of sheets gives rise to an involution of the nonsingular Kummer surface X , which we call “projection” p_α by abuse of notation.
- (3) Switch: The dual surface to X is also a quartic surface which is projectively equivalent to X . Hence there is an involution of X which switches the classes of the nodes and the tropes. We let σ be the switch, which takes N_α to T_α and vice versa.
- (4) Sixteen correlations: These are obtained from projections of the dual quartic surface from its nodes. They have the form $\sigma \circ p_\alpha \circ \sigma$.
- (5) Sixty Cremona transformations called Göpel tetrads. These correspond to sets of four nodes such that no three of them lie on a trope. There are sixty Göpel tetrads. For each of them, there is a corresponding equation of the Kummer surface of the form

$$\begin{aligned} &A(x^2t^2 + y^2z^2) + B(y^2t^2 + x^2z^2) + C(z^2t^2 + x^2y^2) + Dxyzt \\ &+ F(yt + zx)(zt + xy) + G(zt + xy)(xt + yz) + H(xt + yz)(yt + xz) = 0, \end{aligned}$$

and the Cremona transformation is given by $(x, y, z, t) \mapsto (yzt, ztx, txy, xyz)$.

Finally, Keum [Ke2] defined 192 new automorphisms in 1997, and shortly afterward Kondo [Ko] showed that these together with the classical automorphisms generate the automorphism group of X . Note that Kondo’s definition of Keum’s automorphisms is slightly different from the automorphisms defined by Keum (the two are related to each other by multiplication by a classical automorphism). Here, we will use Kondo’s notation, as it seems more convenient. However, we will not explicitly define Keum’s automorphisms here.

Let M be any even hyperbolic lattice. That is, $\langle x, x \rangle \in 2\mathbb{Z}$ for $x \in M$ (which implies $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in M$) and the form $\langle \cdot, \cdot \rangle$ on M has signature $(1, r)$ for some positive integer r . We say $\delta \in M$ is a **root** if $\langle \delta, \delta \rangle = -2$. For each root δ , there is an isometry s_δ of M defined by reflection in the hyperplane H_δ orthogonal to δ :

$$s_\delta(x) = x + \langle x, \delta \rangle \delta.$$

We now recall the general structure of the orthogonal group $O(M)$. Let $P(M)$ be a connected component of the set

$$\{x \in M \otimes \mathbb{R} \mid \langle x, x \rangle > 0\},$$

and $W(M)$, the Weyl group of M , be the subgroup of $O(M)$ generated by all the reflections in the roots of M . Then the hyperplanes H_δ divide $P(M)$ into regions called chambers: these are the connected components of $P(M) \setminus (\bigcup H_\delta)$. These are permuted simply transitively by the Weyl group. If we fix a chamber $D(M)$ and any interior point $\rho \in D(M)$, we obtain a partition of the roots $\Delta = \Delta^+ \sqcup \Delta^-$ into positive and negative roots with respect to ρ or $D(M)$. Let $G(M)$ be the group of (affine) symmetries of $D(M)$. Then $O(M)$ is a split extension of $\{\pm 1\} \cdot W(M)$ by $G(M)$ (see [V] for details).

The lattice $S = NS(X)$ can be primitively embedded into the even unimodular lattice L of signature $(1, 25)$, which is the direct sum $\Lambda \oplus U$ of the Leech lattice and a hyperbolic plane, such that the orthogonal complement T of S in L has its root sublattice isomorphic to $A_3(-1) \oplus A_1(-1)^6$. There is an explicit description of a convex fundamental domain D in $L \otimes \mathbb{R}$ for the reflection group of L due to Conway [Conw]. Following Borchers [B], let $D' = D \cap P(S)$ where $P(S)$ is a connected component of the set $\{x \in S \otimes \mathbb{R} \mid \langle x, x \rangle > 0\}$. In fact, we will choose D' so that it contains the Weyl vector $(\underline{0}, 0, 1) \in \Lambda_{24} \oplus U$.

The region D' is defined by finitely many inequalities. More precisely, to each of the nodes and tropes, the projections and correlations, Cremona transformations and Keum's automorphisms, Lemma 5.1 of [Ko] associates a Leech root $r \in L$. Let r' be its orthogonal projection to $S \otimes \mathbb{Q}$. Then

$$D' = \{x \in S \otimes \mathbb{R} \mid \langle x, r' \rangle \geq 0\}$$

So D' is bounded by $16 + 16 + 16 + 16 + 60 + 192 = 316$, which can be written down explicitly.

Now recall that

$$D(S) = \{x \in P(S) \mid \langle x, \delta \rangle > 0 \text{ for any nodal class } \delta\}$$

is the ample or Kähler cone of X and a fundamental domain with respect to the Weyl group $W(S)$. Let $O(S)$ be the orthogonal group of the lattice S and

$$\Gamma(S) = \{g \in O(S) \mid g|_{A_S}^* = \pm 1\}$$

where $A_S = S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/4\mathbb{Z}$ is the discriminant group of S . Then Theorem 4.1 of Keum [Ke2] shows that

$$\text{Aut}(X) \cong \{g \in \Gamma(S) \mid g \text{ preserves the Kähler cone}\} \cong \Gamma(S)/W(S).$$

We note that the automorphism group $O(q_S)$ of the discriminant form q_S on the discriminant group A_S is isomorphic to an extension of $\mathbb{Z}/2\mathbb{Z}$ by $Sp_4(\mathbb{F}_2) \cong S_6$.

We have that $D' \subset D(S)$, and in fact $D(S)$ contains an infinite number of translates of D' . Let N be the subgroup of $\text{Aut}(X)$ generated by the projections, correlations, Cremona transformations and Keum's automorphisms. We will also think of N as a subgroup of $O(S)$. Also, let K be the finite group of $O(S)$ generated by the translations, switch, and the permutations of the Weierstrass points. Then $K \cong \text{Aut}(D') \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^4 \rtimes Sp_4(\mathbb{F}_2) \cong (\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_6$, a group of order $6! \cdot 2^5$. It is shown in Kondo's paper (this follows directly from Lemma 7.3) that N and K together generate the following group $O(S)^+$, which will play an important role.

$$O(S)^+ = \{g \in O(S) \mid g \text{ preserves the Kähler cone}\}.$$

The above groups fit into the following commutative diagram, with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & W(S) & \longrightarrow & \Gamma(S) & \longrightarrow & \text{Aut}(X) \longrightarrow 1 \\
 & & \parallel & & \downarrow & & \downarrow \\
 1 & \longrightarrow & W(S) & \longrightarrow & O(S) & \longrightarrow & O(S)^+ \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & O(q_S)/\{\pm 1\} & \equiv & O(q_S)/\{\pm 1\} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

with $O(q_S)/\{\pm 1\} \cong Sp_4(\mathbb{F}_2) \cong S_6$.

It follows that D' is like a fundamental domain for the action of N on $D(S)$, in a sense made precise by the following lemma.

Lemma 1. *Let $v \in S$. Then there exists $g \in N$ such that $g(v) \in \overline{D'}$.*

Proof. The proof is analogous to that of Lemma 7.3 of [Ko]. Let w' be the projection of the Weyl vector w , so $w' = 2H - \sum N_\mu/2$. Choose $g \in N$ such that $\langle g(v), w' \rangle$ is minimum possible. For of the chosen generators of N (i.e. a projection, correlation, Cremona transformation or Keum's automorphism), let r be the associated Leech root as above and let ι_r be the isometry of S induced by this automorphism. Let r' be the projection to $S \otimes \mathbb{Q}$ of r .

We have

$$\langle g(v), w' \rangle \leq \langle \iota_r g(v), w' \rangle = \langle g(v), \iota_r^{-1} w' \rangle,$$

where the inequality follows from the choice of g . Now, if ι_r corresponds to a projection or correlation, then ι_r is merely reflection in r' . Therefore,

$$\iota_r^{-1}(w') = \iota_r(w') = w' + 2\langle w', r' \rangle r'.$$

Therefore

$$\langle g(v), w' \rangle \leq \langle g(v), w' \rangle + 2\langle g(v), r' \rangle \langle w', r' \rangle.$$

Since $\langle w', r' \rangle > 0$, we obtain $\langle g(v), r' \rangle \geq 0$.

Next, if ι_r corresponds to a Cremona transformation $c_{\alpha,\beta,\gamma,\delta}$ corresponding to a Göpel tetrad, then $\iota_r = r_{\alpha,\beta,\gamma,\delta} \circ t_{\alpha,\beta,\gamma,\delta}$, where $r_{\alpha,\beta,\gamma,\delta}$ is a reflection with respect to the corresponding r' and $t_{\alpha,\beta,\gamma,\delta}$ fixes $H, N_\alpha, N_\beta, N_\gamma, N_\delta$ and permutes the remaining nodes in a specific manner that we shall not need to describe here.

It follows that

$$\iota_r^{-1}(w') = \iota_r(w') = r_{\alpha,\beta,\gamma,\delta} \circ t_{\alpha,\beta,\gamma,\delta}(w') = r_{\alpha,\beta,\gamma,\delta}(w') = w' + 2\langle w', r' \rangle r'$$

where the middle equality follows because $w' = 2H - \sum N_\mu/2$ is symmetric in the nodes.

As before we obtain $\langle g(v), r' \rangle \geq 0$.

Finally, if ι_r corresponds to a Keum's automorphism, then by an explicit calculation in [Ko], we have

$$\iota_r^{-1}(w') = w' + 8\sigma(r').$$

Hence, we obtain

$$\langle g(v), w' \rangle \leq \langle g(v), w' \rangle + 8\langle g(v), \sigma(r') \rangle$$

or $\langle g(v), \sigma(r') \rangle \geq 0$. As r' ranges over the 192 vectors corresponding to Keum's automorphisms, so does $\sigma(r')$. Therefore, we must have $\langle g(v), r' \rangle \geq 0$ for all r' , and therefore $g(v) \in \overline{D'}$ by definition. \square

The calculations of [Ko] show that N is a normal subgroup of $O(S)^+$, that $K \cap N = \{1\}$, and therefore that $O(S)^+$ is a split extension of K by N . Therefore we have an exact sequence

$$1 \rightarrow N \rightarrow O(S)^+ \rightarrow \text{Aut}(D') \rightarrow 1.$$

Now we have $\text{Aut}(X) \cong \{g \in O(S)^+ | g|_{A_S}^* = \pm 1\}$. We can check that every element of N acts trivially on the discriminant group A_S . Therefore $\text{Aut}(X)$ is the inverse image of the subgroup of $K \cong \text{Aut}(D')$ which acts trivially on the discriminant group. This is easily seen to be the 2-elementary abelian group generated by the translations and the switch. Hence we see that the automorphism group of X is a split extension of this elementary 2-abelian group by N .

In what follows, we will actually work with the entire group $O(S)^+$, since we would like to compute the elliptic fibrations modulo the action of S_6 as well. We will see that including this extra symmetry reduces the number of elliptic fibrations to a more manageable number, namely 25, and does not cause any loss of information, as we may find all the elliptic fibrations by just computing the orbits of these 25 under S_6 .

3.2. Elliptic divisors and a sixteen-dimensional polytope. Following Keum, we will construct a polytope in \mathbb{R}^{16} associated to the domain D' in Lorentzian space, whose vertices will give us information about the elliptic divisors. Writing elements of $S \otimes \mathbb{R}$ as $\beta H - \alpha_0 N_0 - \dots - \alpha_{45} N_{45}$, we see that the quadratic form is $4\beta^2 - 2 \sum \alpha_\mu^2$.

If F is the class of a fiber for a genus 1 fibration, then $F^2 = 0$, and F is nef. Conversely, if $F^2 = 0$ for a divisor (class) on a K3 surface, then an easy application of the Riemann-Roch theorem shows that F or $-F$ must be an effective divisor class. We may move F to the nef cone by an element of $\{\pm 1\} \cdot W(S)$, and then a result from [PSS] establishes that the linear system defined by F gives rise to a genus 1 fibration $X \rightarrow \mathbb{P}^1$. Of course, if $F \in \overline{D}(S)$, then we don't need to move F by an element of the Weyl group at all.

The condition $F^2 = 0$ translates to $\sum (\alpha_\mu / \beta)^2 = 2$. If furthermore we want the fibration to be an elliptic fibration (i.e. to have a section), there must exist an effective divisor $O \in S$ such that $O \cdot F = 1$. If two divisors F and F' are related by an element $g \in \text{Aut}(X)$, then it is clear that they define the “same” elliptic fibration, namely there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow \pi' & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{h} & \mathbb{P}^1 \end{array}$$

where π, π' are elliptic fibrations defined by F, F' respectively, and h is an automorphism of \mathbb{P}^1 . So now we may move $F \in \overline{D}(S)$ to \overline{D}' , by applying an element of N and using Lemma 1 above.

We map D' to a polytope \mathcal{P} in \mathbb{R}^{16} by taking $\beta H - \sum \alpha_\mu N_\mu$ to $(x_\mu)_\mu = (\frac{\alpha_\mu}{\beta})_\mu$. This map just normalizes the coefficient of H to 1 in the expression for a divisor. Therefore the image \mathcal{P} of \overline{D}' is the intersection of a set of $32 + 32 + 60 + 192 = 316$ half-spaces of the form $\sum c_\mu x_\mu \leq d$. An argument of Keum [Ke2] shows that the polytope \mathcal{P} must be contained in the ball of radius $\sqrt{2}$, for otherwise there would be infinitely many rational points of norm 2 in \mathcal{P} , giving rise to infinitely many inequivalent elliptic fibrations and contradicting Sterk's theorem.

In any case, we can enumerate all the vertices of \mathcal{P} using the program SymPol [RS], or linear programming software such as `glpk`. Here symmetry plays an important role, as we can restrict ourselves to computing the vertices modulo the symmetries of D' , which reduces the number by a factor of approximately $32 \cdot 720 = 23040$. The details of these calculations are described in Appendix B, which also describes how to verify that these calculations are correct; this verification is much simpler than finding the vertices in the first place! We find that the polytope \mathcal{P} has 1492 vertices modulo the symmetries of D' (note that the symmetries of D' are not always affine linear symmetries of \mathcal{P} , even though they are linear transformations in \mathbb{R}^{17}). Out of these 54 vertices have norm 2, giving rise to at most 54 distinct elliptic divisors modulo $N \rtimes \text{Aut}(D') = O(S)^+$.

Recall that the subgroup $\text{Aut}(X)$ of $O(S)^+$ is normal, the quotient being isomorphic to S_6 (and in fact, there is a complementary subgroup, induced by the permutations of the Weierstrass points). This is because within

$\text{Aut}(D')$, the subgroup $(\mathbb{Z}/2\mathbb{Z})^5$ generated by translations and a switch is normalized by its complementary subgroup S_6 , and is therefore normal. Hence the inverse image $\text{Aut}(X)$ of $(\mathbb{Z}/2\mathbb{Z})^5$ under $O(S)^+ \rightarrow \text{Aut}(D')$ is normal in $O(S)^+$. Here we will just compute the elliptic divisors up to the action of $O(S)^+$. This set turns out to be smaller than the set of vertices of \mathcal{P} of norm 2 modulo its symmetries, because there are some extra symmetries coming from N which connect some of the vertices.

The following table lists the elliptic divisors which are obtained from the vertices of \mathcal{P} , up to symmetry. We use the shorthand notation

$$(c_0, c_1, \dots, c_5, c_{12}, \dots, c_{15}, c_{23}, \dots, c_{25}, c_{34}, c_{35}, c_{45}, d)$$

to represent the divisor class $\sum c_\mu N_\mu + dH$. For a given elliptic divisor D , it is easy to check if it gives an elliptic fibration with a section: we check that the divisor D is primitive in S , and then check whether there is an element of S whose intersection with D is 1. If this condition fails, then D will only provide a genus one fibration.

Elliptic divisor	Section	Number
$(-1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	Yes	1
$(-2, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2)$	Yes	2
$(-2, -1, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 2)$	Yes	3
$(-2, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, -1, 0, 0, 2)$	Yes	4
$(-2, -1, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 2)$	No	
$(-3, -3, -1, -1, 0, 0, -1, -1, 0, 0, -1, -2, 0, -2, 0, -1, 4)/2$	Yes	5
$(-4, -2, -1, 0, 0, -1, -1, 0, 0, -1, -1, -1, 0, -2, -1, -1, 4)/2$	Yes	6
$(0, -1, 0, -1, -1, -1, 0, -1, 0, 0, -1, 0, -1, -1, 0, 0, 2)$	No	
$(-1, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, 2)/2$	Yes	7
$(-3, -2, -1, 0, 0, 0, 0, 0, -1, 0, -1, -1, 0, -1, 0, 0, 3)$	Yes	8
$(-2, -2, 0, 0, 0, -1, 0, 0, 0, -1, 0, -2, -2, 0, 0, 0, 3)$	Yes	8A
$(-3, -2, -1, 0, 0, 0, 0, 0, 0, -1, -1, -1, 0, -1, 0, 0, 3)$	Yes	9
$(-2, -2, -1, 0, 0, -1, 0, -1, -1, 0, -2, 0, -1, -1, 0, 0, 3)/2$	Yes	10
$(-2, -1, -1, -1, -1, 0, -1, 0, 0, 0, 0, 0, -2, -1, 0, -2, 3)$	No	
$(0, -1, -1, 0, 0, 0, -1, -1, -1, 0, -2, -2, 0, -1, 0, -2, 3)$	No	
$(-3, -2, 0, 0, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 3)$	Yes	11
$(-3, -2, 0, 0, 0, -1, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 3)$	No	
$(-2, -2, 0, 0, 0, -2, 0, -1, 0, 0, 0, -1, -2, 0, 0, 0, 3)$	Yes	11A
$(-5, -4, 0, 0, 0, -3, -1, -1, -1, 0, -1, -3, -2, -1, 0, -2, 6)/2$	Yes	12
$(-6, -3, 0, 0, -3, 0, -1, -1, 0, -1, -2, -1, -2, -1, -2, -1, 6)/2$	Yes	13
$(0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -2, -2, -1, -1, 0, -2, 3)$	No	
$(-3, -1, 0, 0, -1, -1, -1, -1, 0, 0, 0, -1, -1, -1, -1, 0, 3)/2$	Yes	14
$(-4, -2, 0, 0, 0, -2, 0, 0, -1, -1, 0, -2, -1, -1, 0, 0, 4)$	Yes	15
$(-3, -2, 0, 0, -1, -2, 0, -1, 0, 0, 0, -2, -3, 0, 0, 0, 4)$	No	
$(-3, -3, 0, 0, 0, -2, 0, -1, 0, -1, 0, -2, -2, 0, 0, 0, 4)$	Yes	15A
$(-4, -2, 0, 0, 0, -2, -1, 0, -1, 0, 0, -2, -1, -1, 0, 0, 4)$	Yes	16
$(-3, -3, 0, 0, 0, -2, 0, 0, 0, 0, -1, -2, -2, 0, 0, -1, 4)$	No	
$(-4, -2, 0, 0, -1, -1, 0, -1, 0, 0, 0, -2, -2, 0, 0, -1, 4)$	Yes	17
$(-3, -3, 0, 0, 0, -2, -1, -1, 0, 0, 0, -2, -2, 0, 0, 0, 4)$	Yes	18
$(-4, -2, 0, 0, 0, -2, 0, 0, 0, 0, -1, -2, -1, -1, 0, -1, 4)$	No	
$(-4, -2, 0, 0, 0, -2, 0, 0, -1, 0, -1, -2, -1, -1, 0, 0, 4)$	Yes	18A
$(-3, -3, -2, 0, 0, 0, 0, 0, 0, -2, -1, -1, -1, -1, -1, -1, 4)/2$	Yes	19

Elliptic divisor	Section	Number
$(-3, -3, 0, 0, 0, -2, -1, 0, 0, 0, 0, -2, -2, 0, 0, -1, 4)$	Yes	16A
$(-4, -1, 0, -1, 0, -2, 0, 0, -1, 0, 0, -2, -2, -1, 0, 0, 4)$	No	
$(-4, -2, 0, 0, 0, -2, -1, 0, -1, 0, 0, -2, -1, 0, 0, -1, 4)$	No	
$(-4, -4, 0, 0, -2, 0, 0, -1, 0, -1, -2, -2, -2, 0, 0, 0, 5)$	Yes	20A
$(-5, -2, 0, -1, 0, -2, 0, 0, -1, -1, 0, -3, -2, -1, 0, 0, 5)$	Yes	20
$(-1, 0, -2, 0, 0, -1, -3, -5, 0, -1, 0, -2, -2, 0, 0, -1, 5)$	No	
$(-4, -3, 0, 0, -1, -2, 0, -2, -1, 0, -1, -2, -3, 0, -1, 0, 5)/2$	Yes	21
$(-4, -3, 0, 0, -1, -2, 0, -1, 0, -1, 0, -3, -3, 0, 0, 0, 5)$	Yes	20B
$(-5, -2, 0, -1, 0, -2, 0, -1, -2, 0, -1, -2, -2, -1, -1, 0, 5)/2$	Yes	22
$(-5, -2, 0, 0, -2, -1, 0, -1, -1, 0, -3, 0, -2, -1, 0, 0, 5)$	Yes	23
$(-1, -3, 0, 0, 0, -2, -4, -1, 0, -5, 0, -4, -5, 0, 0, -1, 7)$	No	
$(-3, -5, -3, -2, 0, -1, 0, 0, 0, -6, 0, 0, 0, -2, -1, -3, 7)$	No	
$(-1, -1, -3, -2, -3, -4, 0, -1, 0, -7, -2, 0, -2, 0, 0, 0, 7)$	No	
$(-3, -4, 0, -5, -2, 0, -1, 0, 0, 0, -1, 0, -1, -5, 0, -4, 7)$	No	
$(-1, 0, -6, -3, -3, -1, 0, -2, 0, -3, -2, 0, -5, 0, 0, 0, 7)$	No	
$(-6, -5, 0, 0, -3, 0, 0, -1, 0, -2, -3, -3, -2, -1, 0, 0, 7)$	Yes	24
$(-2, 0, 0, -1, -1, -2, -7, -3, 0, -2, -1, 0, 0, -3, -4, 0, 7)$	No	
$(-4, -5, 0, 0, 0, -5, 0, -3, -1, 0, 0, -2, -4, -1, -1, 0, 7)$	Yes	24A
$(-2, -1, -7, 0, 0, -4, 0, -2, 0, -3, -1, 0, 0, -1, -2, -3, 7)$	No	
$(-6, -4, 0, 0, -2, -2, 0, -2, -1, -1, 0, -4, -4, 0, 0, 0, 7)$	Yes	24B
$(-3, -2, -4, 0, -3, -1, 0, -1, 0, -1, -7, 0, 0, 0, -2, -2, 7)$	Yes	24C
$(-7, -5, 0, -2, -2, 0, 0, 0, 0, -4, -3, -3, -3, -1, -1, -1, 8)/2$	Yes	25

Note that the elliptic divisors we will ultimately use to compute with each fibration are not always the ones listed above, but are related by an element of $O(S)^+$. Fibrations with the same numerical label, such as 8 and 8A, are equivalent, and we show this in the analysis. The fibrations with different numerical labels are inequivalent: they are easily distinguished by having different configurations of reducible fibers.

3.3. Results. The analysis of the previous section leads to a (computer-assisted) proof of our main theorem.

Theorem 2. *There are exactly 25 distinct elliptic fibrations with section on a generic Jacobian Kummer surface $X = \text{Km}(J(C))$, for a genus 2 curve C over an algebraically closed field of characteristic 0, modulo the action of $\text{Aut}(X)$ and the permutations S_6 of the Weierstrass points of C .*

We list the Kodaira-Néron types of the reducible fibers, the torsion subgroups and Mordell-Weil lattices of each of the twenty-five fibrations in the table below. In the remainder of the paper, we will analyze each of the fibrations. For background on elliptic surfaces, we refer the reader to [Sh2].

Let A_n, D_n, E_6, E_7, E_8 be the usual root lattices, and for a lattice M , let M^* be its dual. The symbol $\langle \alpha \rangle$ denotes the one-dimensional lattice generated by an element of norm α .

Let P_n be lattice whose Gram matrix is the $n \times n$ matrix with 3's on the diagonal and 1's off the diagonal. The minimal norm of P_n is 3, and its discriminant (the square of its covolume) is easily seen to be $2^{n-1}(n+2)$. It is an easy exercise to check that the inverse of this Gram matrix (i.e. a Gram matrix for the dual lattice

P_n^*) is $1/(2n+4)$ times the matrix which has $n+1$'s on the diagonal and -1 's off the diagonal.

Fibration	Reducible Fibers	Torsion subgroup	Mordell-Weil lattice
1	$D_4^2 A_1^6$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\langle 1 \rangle$
2	$D_6 D_4 A_1^4$	$\mathbb{Z}/2\mathbb{Z}$	$\langle 1 \rangle$
3	$D_6 A_3 A_1^6$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	0
4	$D_4^3 A_3$	$\mathbb{Z}/2\mathbb{Z}$	0
5	$A_7 A_3^2$	$\mathbb{Z}/2\mathbb{Z}$	$P_2[\frac{1}{2}]$
6	$A_7 A_3 A_1^2$	$\mathbb{Z}/2\mathbb{Z}$	$A_3^*[2]$
7	A_3^4	$\mathbb{Z}/2\mathbb{Z}$	$\langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$
8	$D_6^2 A_1^2$	0	$\langle 1 \rangle$
9	$D_6 D_5 A_1^4$	$\mathbb{Z}/2\mathbb{Z}$	0
10	$A_5^2 A_1^2$	$\mathbb{Z}/2\mathbb{Z}$	$(A_2 \oplus A_1)[\frac{2}{3}]$
11	$D_8 A_1^6$	$\mathbb{Z}/2\mathbb{Z}$	$\langle 1 \rangle$
12	$E_6 D_5$	0	$P_4^*[4]$
13	$E_6 D_4$	0	$A_5^*[2]$
14	A_5^2	0	$A_2[\frac{2}{3}] \oplus A_2^*[2] \oplus \langle 1 \rangle$
15	$D_8 D_4 A_3$	0	0
16	$E_7 D_4 A_1^3$	0	$\langle 1 \rangle$
17	$D_7 D_4^2$	0	0
18	$E_7 A_3 A_1^5$	$\mathbb{Z}/2\mathbb{Z}$	0
19	D_5^2	0	$A_3^*[2] \oplus P_2[\frac{1}{2}]$
20	$D_{10} A_1^4$	0	$\langle 1 \rangle$
21	$A_9 A_1^3$	$\mathbb{Z}/2\mathbb{Z}$	$P_3^*[4]$
22	$A_9 A_1$	0	$(A_4^* \oplus A_1^*)[2]$
23	$D_9 A_1^6$	$\mathbb{Z}/2\mathbb{Z}$	0
24	$E_8 A_1^6$	0	$\langle 1 \rangle$
25	D_9	0	$P_6^*[4]$

In the analysis of the fibrations, we will describe explicit generators for the Mordell-Weil group and give the height pairing matrix for a basis of the Mordell-Weil lattice. We shall omit the lattice basis reduction step which shows that the Mordell-Weil lattice is as listed above, since this step is easily performed with a computer algebra package such as PARI/gp.

3.4. Formulas for the Kummer surface. The curve C is a double cover of \mathbb{P}^1 branched at six Weierstrass points. Since k is algebraically closed, without loss of generality we may put three of the Weierstrass points at $0, 1, \infty$. We assume from now on that the Weierstrass equation of the genus 2 curve C is given by

$$y^2 = x(x-1)(x-a)(x-b)(x-c)$$

We may use the formulas from [CF] to write down the equation of the Kummer quartic surface. This is given by

$$(3.1) \quad K_2 z_4^2 + K_1 z_4 + K_0 = 0$$

with

$$\begin{aligned}
K_2 &= z_2^2 - 4 z_1 z_3 \\
K_1 &= (-2 z_2 + 4(a + b + c + 1) z_1) z_3^2 \\
&\quad + (-2(bc + ac + c + ab + b + a) z_1 z_2 + 4(ab c + bc + ac + ab) z_1^2) z_3 \\
&\quad - 2abc z_1^2 z_2 \\
K_0 &= z_3^4 - 2(bc + ac + c + ab + b + a) z_1 z_3^3 \\
&\quad + (4(ab c + bc + ac + ab) z_1 z_2 \\
&\quad + (a^2 + b^2 + c^2 - 2ab(a + b + 1) - 2bc(b + c + 1) - 2ac(a + c + 1) \\
&\quad + a^2 b^2 + b^2 c^2 + a^2 c^2 - 2abc(a + b + c + 4)) z_1^2) z_3^2 \\
&\quad + (-4abc z_1 z_2^2 + 4abc(c + b + a + 1) z_1^2 z_2 \\
&\quad - 2abc(bc + ac + c + ab + b + a) z_1^3) z_3 \\
&\quad + a^2 b^2 c^2 z_1^4
\end{aligned}$$

We can also work out the locations of the nodes and the equations of the tropes in projective 3-space. These are as follows:

$n_0 = [0 : 0 : 0 : 1]$	$T_o = [1 : 0 : 0 : 0]$
$n_1 = [0 : 1 : 0 : 0]$	$T_1 = [0 : 0 : 1 : 0]$
$n_2 = [0 : 1 : 1 : 1]$	$T_2 = [1 : -1 : 1 : 0]$
$n_3 = [0 : 1 : a : a^2]$	$T_3 = [a^2 : -a : 1 : 0]$
$n_4 = [0 : 1 : b : b^2]$	$T_4 = [b^2 : -b : 1 : 0]$
$n_5 = [0 : 1 : c : c^2]$	$T_5 = [c^2 : -c : 1 : 0]$
$n_{12} = [1 : 1 : 0 : abc]$	$T_{12} = [-abc : 0 : -1 : 1]$
$n_{13} = [1 : a : 0 : bc]$	$T_{13} = [-bc : 0 : -a : 1]$
$n_{14} = [1 : b : 0 : ca]$	$T_{14} = [-ca : 0 : -b : 1]$
$n_{15} = [1 : c : 0 : ab]$	$T_{15} = [-ab : 0 : -c : 1]$
$n_{23} = [1 : a + 1 : a : a(b + c)]$	$T_{23} = [-a(b + c) : a : -(a + 1) : 1]$
$n_{24} = [1 : b + 1 : b : b(c + a)]$	$T_{24} = [-b(c + a) : b : -(b + 1) : 1]$
$n_{25} = [1 : c + 1 : c : c(a + b)]$	$T_{25} = [-c(a + b) : c : -(c + 1) : 1]$
$n_{34} = [1 : a + b : ab : ab(c + 1)]$	$T_{34} = [-ab(c + 1) : ab : -(a + b) : 1]$
$n_{35} = [1 : a + c : ca : ac(b + 1)]$	$T_{35} = [-ca(b + 1) : ca : -(c + a) : 1]$
$n_{45} = [1 : b + c : bc : bc(a + 1)]$	$T_{45} = [-bc(a + 1) : bc : -(b + c) : 1]$

Here we have displayed the equations of the hyperplanes defining the tropes in terms of coordinates on the dual projective space. For instance, we can read out from the list above that the trope T_3 is given by $a^2 z_1 - a z_2 + z_3 = 0$.

3.5. Igusa-Clebsch invariants. For later reference, we recall the definition and properties of the Igusa-Clebsch invariants [Cl, I, M] of a genus 2 curve C . These are actually classical covariants of a sextic form, which transform appropriately under the action of $PGL_2(k)$.

Given a sextic Weierstrass equation for C ,

$$y^2 = f(x) = \sum_{i=0}^6 f_i x^i = f_6 \prod_{i=1}^6 (x - \alpha_i),$$

the first Igusa-Clebsch invariant of f is defined to be

$$I_2(f) = f_6^2 \sum (12)^2 (34)^2 (56)^2 := f_6^2 \sum (\alpha_{i_1} - \alpha_{i_2})^2 (\alpha_{i_3} - \alpha_{i_4})^2 (\alpha_{i_5} - \alpha_{i_6})^2$$

where the sum is over all partitions $\{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\}$ of $\{1, 2, 3, 4, 5, 6\}$ into three subsets of two elements each. Similarly, we define

$$\begin{aligned} I_4(f) &= f_6^4 \sum (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 \\ I_6(f) &= f_6^6 \sum (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 (14)^2 (25)^2 (36)^2 \\ I_{10}(f) &= f_6^{10} \prod (ij)^2. \end{aligned}$$

Note that $I_{10}(f)$ is the discriminant of the sextic polynomial f .

These transform covariantly under the action of $GL_2(k)$: the sextic f transforms under the action of $g \in GL_2(k)$ with

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the polynomial

$$(g \cdot f)(x) = f\left(\frac{ax+b}{cx+d}\right) (cx+d)^6$$

and the Igusa-Clebsch invariants transform as

$$I_d(g \cdot f) = \det(g)^{-3d} I_d(f)$$

It is easy to see that there is a unique way to extend the Igusa-Clebsch invariants to the case when f is a quintic (i.e. one of the Weierstrass points of the genus two curve is at infinity) so that they continue to satisfy the transformation property above.

For the Igusa-Clebsch invariants of the Weierstrass equation

$$y^2 = x(x-1)(x-a)(x-b)(x-c),$$

we obtain the following expressions.

$$\begin{aligned} I_2 &= 2 \left(3\sigma_1^2 - 2(\sigma_2 + 4\sigma_3)\sigma_1 + 3\sigma_2^2 - 8\sigma_2 + 12\sigma_3 \right) \\ I_4 &= 4 \left(-3\sigma_3\sigma_1^3 + (\sigma_2^2 - \sigma_3\sigma_2 + \sigma_3^2 + 3\sigma_3)\sigma_1^2 + (-\sigma_2^2 + 11\sigma_3\sigma_2 - 3\sigma_3)\sigma_1 \right. \\ &\quad \left. - 3\sigma_2^3 + (3\sigma_3 + 1)\sigma_2^2 - 3\sigma_3^2\sigma_2 - 18\sigma_3^2 \right) \\ I_6 &= 2 \left(-12\sigma_3\sigma_1^5 + 2(2\sigma_2^2 + 5\sigma_3\sigma_2 + 12\sigma_3^2 + 6\sigma_3)\sigma_1^4 \right. \\ &\quad + (-4\sigma_2^3 - 2(9\sigma_3 + 2)\sigma_2^2 + (10\sigma_3 + 59)\sigma_3\sigma_2 - 4(3\sigma_3^2 + 17\sigma_3 + 3)\sigma_3)\sigma_1^3 \\ &\quad + (4\sigma_2^4 - 2(2\sigma_3 + 9)\sigma_2^3 + 4(\sigma_3^2 + 1)\sigma_2^2 - (97\sigma_3 + 33)\sigma_3\sigma_2 + 16\sigma_3^3 + 5\sigma_3^2)\sigma_1^2 \\ &\quad + (10\sigma_2^4 + (59\sigma_3 + 10)\sigma_2^3 - (33\sigma_3 + 97)\sigma_3\sigma_2^2 \\ &\quad + 2(19\sigma_3^2 + 103\sigma_3 + 19)\sigma_3\sigma_2 + 3(25\sigma_3 - 7)\sigma_3^2)\sigma_1 \\ &\quad - 12\sigma_2^5 + 12(\sigma_3 + 2)\sigma_2^4 - 4(3\sigma_3^2 + 17\sigma_3 + 3)\sigma_2^3 \\ &\quad \left. + (5\sigma_3 + 16)\sigma_3\sigma_2^2 - 3(7\sigma_3 - 25)\sigma_3^2\sigma_2 - 18(\sigma_3^2 + 7\sigma_3 + 1)\sigma_3^2 \right) \\ I_{10} &= a^2 b^2 c^2 (a-1)^2 (b-1)^2 (c-1)^2 (a-b)^2 (b-c)^2 (c-a)^2. \end{aligned}$$

Here, $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric polynomials in a, b, c .

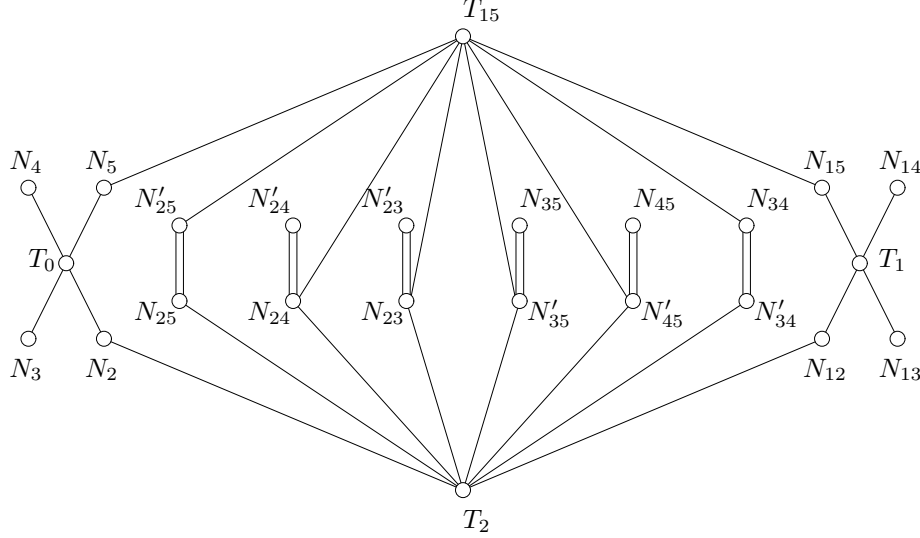
$$\begin{aligned} \sigma_1 &= a + b + c \\ \sigma_2 &= ab + bc + ca \\ \sigma_3 &= abc. \end{aligned}$$

We will also use another expression I_5 which is not quite an invariant: it is a square root of the discriminant I_{10} :

$$I_5 = abc(a-1)(b-1)(c-1)(a-b)(b-c)(c-a).$$

4. FIBRATION 1

This corresponds to the divisor class $H - N_0 - N_1 = 2T_0 + N_2 + N_3 + N_4 + N_5$. We can see some of the classes of the sections and the reducible components of the reducible fibers in the Dynkin diagram below. As usual, the nodes represent roots in the Néron-Severi lattice, which correspond to smooth rational curves of self-intersection -2 . In addition, we label the non-identity component of the fiber containing N_{23} by N'_{23} , etc.: this is just convenient (non-standard) notation.



This elliptic fibration has two D_4 fibers, $2T_0 + N_2 + N_3 + N_4 + N_5$ and $2T_1 + N_{12} + N_{13} + N_{14} + N_{15}$, as well as six A_1 fibers as shown in the diagram. We take the identity section to be T_2 , and observe that T_3, T_4, T_5 are 2-torsion sections. Therefore this elliptic fibration has full 2-torsion. The trivial lattice (that is, the sublattice of $\text{NS}(X)$ spanned by the classes of the zero section and the irreducible components of fibers) has rank $2 + 2 \cdot 4 + 6 \cdot 1 = 16$ and discriminant $2^6 \cdot 4^2 = 2^{10}$. A non-torsion section is given by T_{12} ; its translations by the torsion sections T_3, T_4, T_5 are the sections T_{13}, T_{14}, T_{15} respectively. The height of any of these four non-torsion sections is $4 - (1 + 1) - (1/2 + 1/2) = 1$. Therefore we see that the sublattice of $\text{NS}(X)$ generated by the trivial lattice, the torsion sections, and any of these four non-torsion sections has rank 17 and absolute discriminant $2^{10} \cdot (1/2)^2 \cdot 1 = 2^6$. Consequently, it must equal all of $\text{NS}(\text{Km}(J(C)))$, which we know to be of the same rank and discriminant. Therefore, the Mordell-Weil rank for this elliptic fibration is 1.

Now we describe a Weierstrass equation for this elliptic fibration.

First, the fact that the class of the fiber is $H - N_0 - N_1$ indicates that we must look for linear polynomials in z_1, \dots, z_4 vanishing on the singular points $n_0 = [0 : 0 : 0 : 1]$ and $n_1 = [0 : 1 : 0 : 0]$ of the quartic Kummer surface with equation 3.1. Clearly, a basis for this space is given by the monomials z_1 and z_3 . Therefore, we set the elliptic parameter $t = z_3/z_1$ and substitute for z_3 in equation 3.1. Simplifying the resulting (quadratic in z_4) equation, we get the following form

$$\eta^2 = 4t(\xi - t - 1)(a\xi - t - a^2)(b\xi - t - b^2)(c\xi - t - c^2)$$

with

$$\begin{aligned} \xi &= z_2/z_1 \\ \eta &= (z_4/z_1)(\xi^2 - 4t) - \xi(t^2 + (a + b + c + ab + bc + ca)t + abc) \\ &\quad + 2t((a + b + c + 1)t + (ab + bc + ca + abc)) \end{aligned}$$

This is the equation of a genus 1 curve over $k(t)$, which clearly has rational points. For instance, there are four Weierstrass points given by $\xi = 1 + t, (a^2 + t)/a, (b^2 + t)/b, (c^2 + t)/c$. Translating by the first of these, we obtain the Weierstrass equation

$$(4.1) \quad \begin{aligned} y^2 = & (x + 4(a-1)(b-1)ct(t-a)(t-b)) \cdot \\ & (x + 4(b-1)(c-1)at(t-b)(t-c)) \cdot \\ & (x + 4(c-1)(a-1)bt(t-c)(t-a)) \end{aligned}$$

with

$$\begin{aligned} x &= \frac{4(a-1)(b-1)(c-1)t(t-a)(t-b)(t-c)}{\xi - 1 - t} \\ y &= \frac{4(a-1)(b-1)(c-1)t(t-a)(t-b)(t-c)\eta}{(\xi - 1 - t)^2}. \end{aligned}$$

This elliptic fibration was the one described in [Sh1].

Now it is easy to check that the elliptic fibration given by equation (4.1) has D_4 fibers at $t = 0$ and ∞ , and six A_1 fibers at $t = a, b, c, ab, bc, ca$. Using the change of coordinates and the equations of the nodes and tropes on the Kummer surface, we may compute the components of the reducible fibers, and also some torsion and non-torsion sections of the elliptic fibration.

The reducible fibers are as follows:

Position	Reducible fiber	Kodaira-Néron type
$t = \infty$	$N_2 + N_3 + N_4 + N_5 + 2T_0$	D_4
$t = 0$	$N_{12} + N_{13} + N_{14} + N_{15} + 2T_1$	D_4
$t = a$	$N_{23} + N'_{23}$	A_1
$t = b$	$N_{24} + N'_{24}$	A_1
$t = c$	$N_{25} + N'_{25}$	A_1
$t = ab$	$N_{34} + N'_{34}$	A_1
$t = bc$	$N_{45} + N'_{45}$	A_1
$t = ca$	$N_{35} + N'_{35}$	A_1

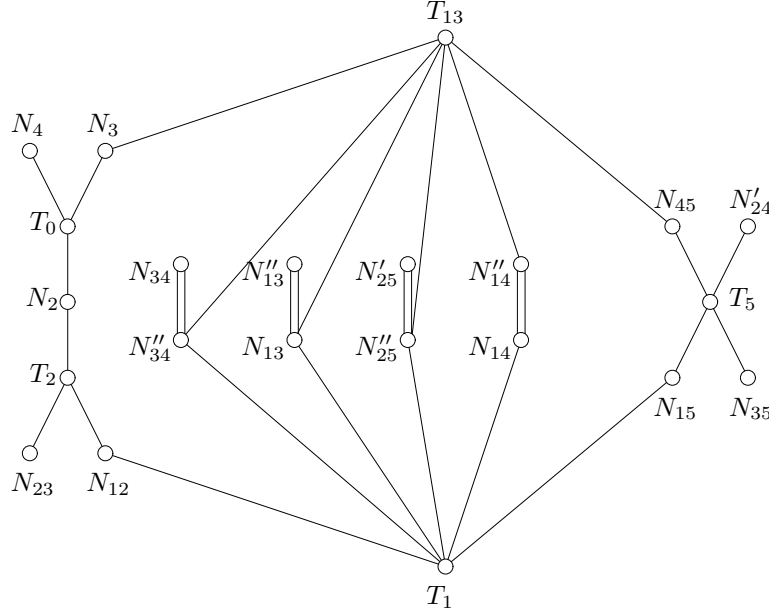
Equations for some of the torsion and non-torsion section are given in the tables below.

Torsion section	Equation
T_2	$x = y = \infty$
T_3	$x = -4a(b-1)(c-1)t(t-b)(t-c), \quad y = 0$
T_4	$x = -4b(c-1)(a-1)t(t-c)(t-a), \quad y = 0$
T_5	$x = -4c(a-1)(b-1)t(t-a)(t-b), \quad y = 0$

Non-torsion section	Equation
T_{12}	$x = 4(t-a)(t-b)(t-c)(t-abc)$ $y = 8(t-a)(t-b)(t-c)(t-ab)(t-ac)(t-bc)$
T_{13}	$x = -4(b-1)(c-1)t(t-a)(at-bc)$ $y = -8(b-1)(c-1)(b-a)(c-a)t^2(t-a)(t-bc)$
T_{14}	$x = -4(c-1)(a-1)t(t-b)(bt-ca)$ $y = -8(c-1)(a-1)(c-b)(a-b)t^2(t-b)(t-ca)$
T_{15}	$x = -4(a-1)(b-1)t(t-c)(ct-ab)$ $y = -8(a-1)(b-1)(a-c)(b-c)t^2(t-c)(t-ab)$

5. FIBRATION 2

This fibration corresponds to the divisor class $N_3 + N_4 + 2T_0 + 2T_2 + 2N_2 + N_{12} + N_{23}$, a fiber of type D_6 .



We compute that the other reducible fibers consist of one D_4 fiber and four A_1 fibers, as in the figure above. We take T_1 to be the zero section, and note that T_4 is a 2-torsion section, whereas T_{13} is a non-torsion section. The height of this section is $4 - 6/4 - 1/2 - 1 = 1$, and therefore the span of these sections and the trivial lattice has rank $1 + 2 + 6 + 4 + 4 = 17$ and discriminant $4 \cdot 4 \cdot 2^4 \cdot 1/2^2 = 64$. So it must be the full Néron-Severi lattice $\text{NS}(X)$.

To obtain the Weierstrass equation of this elliptic fibration, we use a 2-neighbor step from fibration 1, as follows. We compute explicitly the space of sections of the line bundle $\mathcal{O}_X(F_2)$ where $F_2 = 2T_0 + 2T_2 + 2N_2 + N_3 + N_4 + N_{12} + N_{23}$ is the class of the fiber we are considering. The space of sections is 2-dimensional, and the ratio of two linearly independent global sections will be an elliptic parameter for X , for which the class of the fiber will be F_2 . Notice that any global section has a pole of order at most 2 along T_2 , the zero section of fibration 1. Also, it has at most a double pole along N_2 , which is the identity component of the $t = \infty$ fiber, a simple pole along N_{12} , the identity component of the $t = 0$ fiber, and a simple pole along N_{23} , the identity component of the $t = a$ fiber. We deduce that a global section of $\mathcal{O}_X(F_2)$ must have the form

$$\frac{cx + a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4}{t(t-a)}$$

It is clear that 1 is a global section of $\mathcal{O}_X(F_2)$, so we may subtract a term $a_2t(t-a)$ from the numerator, and thereby assume $a_2 = 0$. With this constraint, we will obtain a 1-dimensional space of sections, and the generator of this space will give us a parameter on the base for fibration 2. To pin down the remaining section up to scaling, we may fix the scaling $c = 1$.

To obtain further conditions on the a_i , we look at the order of vanishing along the non-identity components of the fibers at $t = 0, a, \infty$. After some calculation, we find that an elliptic parameter is given by

$$w = \frac{x + 4(a-1)(b-1)ct^2(t-a)}{t(t-a)}$$

If we solve for x in terms of w and substitute in to right hand side of the Weierstrass equation of fibration 1, after simplifying and dividing out some square factors (which we can absorb into y^2) we obtain an equation for fibration 2 in the form $y^2 = q_4(t)$, where q_4 is a quartic in t with coefficients in $\mathbb{Q}(a, b, c, w)$. But in fact, the quartic factors into t times a linear factor in t and a quadratic factor in t . We may write $t = 1/u$ and

dividing y^2 by t^4 , get a cubic equation $y^2 = c_3(u)$. Finally, we can make admissible changes to the Weierstrass equation, and also a linear fractional transformation of the base parameter w , to get the following Weierstrass equation for fibration 2.

$$\begin{aligned} Y^2 = X \bigg(& X^2 + T X \left(4(a-1)bc(c-b)T^2 - 4(2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - bc \right. \\ & - a^2bc - a^2c + 2ac - ab^2 + 2a^2b - ab)T + 4(b-1)(c-a)(ac+b-2a) \bigg) \\ & + T^2 \left(-16(a-1)ab(b-a)(c-1)c(c-b)T^3 + 16a(b-a)(c-1)(b^2c^2 + c^2 \right. \\ & + 2abc^2 - 3bc^2 - ac^2 - 3ab^2c + 2b^2c - a^2bc + 3abc - bc + a^2b^2 - ab^2)T^2 \\ & - 16a(b-1)(b-a)(c-1)(c-a)(2bc+ac-2c-2ab+b)T \\ & \left. + 16a(b-1)^2(b-a)(c-1)(c-a)^2 \right) \bigg) \end{aligned}$$

The new parameters T, X, Y are related to the ones from fibration 1 by the following equations.

$$\begin{aligned} T &= \frac{x + 4(a-1)(b-1)ct(t-a)(t-b)}{4(a-1)bc(c-b)t(t-a)} \\ X &= \frac{-a(x+\mu_1)(x+\mu_2)(x+\mu_3)}{16(a-1)^2b^2c^2(c-b)^2t^4(t-a)^3} \\ Y &= \frac{ay(x+\mu_2)(x+\mu_3)(x+\mu_4)}{64(a-1)^3b^3c^3(c-b)^3t^6(t-a)^4} \end{aligned}$$

with

$$\begin{aligned} \mu_1 &= 4(a-1)b(c-1)t(t-a)(t-c) \\ \mu_2 &= 4(a-1)(b-1)ct(t-a)(t-b) \\ \mu_3 &= 4(b-1)ct(t-a)(at-t-bc+b) \\ \mu_4 &= 4(a-1)ct(t-a)(bt-t-bc+b) \end{aligned}$$

In the following sections, we will simply display the final elliptic parameter and the final Weierstrass equation for each fibration, but leave out the transformation of coordinates (and the intermediate cleaning up of the Weierstrass equation, including any linear fractional transformations of the elliptic parameter). If Fibration n is obtained by a 2-neighbor step from Fibration m , we will write the elliptic parameter t_n as a rational function of t_m, x_m, y_m . Then we will display the Weierstrass equation connecting y_n and x_n (with coefficients polynomials in t_n), except that we will drop the subscripts and simply use the variables t, x, y for convenience. Also, we will give few details about the 2-neighbor step which carries one elliptic fibration into another: this is accomplished as above, by computing the space of sections of the linear system described by the class of the new fiber. The ratio of two linearly independent sections gives an elliptic parameter, and substituting for x or y in terms of the new parameter w gives an equation of the form

$$z^2 = \text{quartic}(t)$$

with coefficients in $\mathbb{Q}(a, b, c, w)$. One then finds an explicit point on this curve. Its existence is guaranteed by the fact that we have a genuine elliptic fibration with a section, and to actually find a point, it is enough to guess a curve on the original fibration whose intersection with the fiber is 1, and compute its new coordinate z using the change of coordinates so far. Finally, the rational point may be used to transform the equation into a standard cubic Weierstrass form. A brief description of the method, with some details, is given in Appendix A and will be omitted in the main body of the paper.

We return to the description of Fibration 2. Its reducible fibers are as follows.

Position	Reducible fiber	Kodaira-Néron type
$t = \infty$	$2T_0 + 2T_2 + 2N_2 + N_3 + N_4 + N_{12} + N_{23}$	D_6
$t = 0$	$2T_5 + N_{15} + N_{35} + N_{45} + N'_{24}$	D_4
$t = 1$	$N_{14} + N''_{14}$	A_1
$t = \frac{c-a}{c}$	$N''_{34} + N_{34}$	A_1
$t = \frac{b-1}{b}$	$N''_{25} + N'_{25}$	A_1
$t = \frac{(b-1)(c-a)}{(a-1)(c-b)}$	$N_{13} + N''_{13}$	A_1

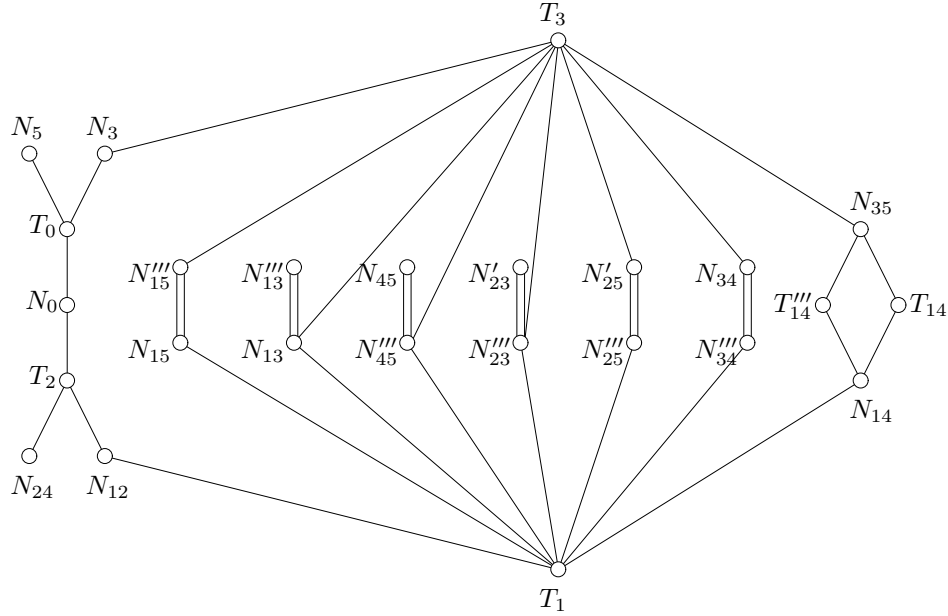
Some of the sections are indicated in the following table.

Section	Equation
T_1	$x = y = \infty$
T_4	$x = y = 0$
T_{13}	$x = 4a(b-a)(c-1)t^2$ $y = -8a(b-1)(b-a)(c-1)(c-a)t^2(t-1)$
T_{15}	$x = -4(bt-b+1)(ct-c+a)((a-1)(c-b)t-(b-1)(c-a))$ $y = -8(b-1)(c-a)(t-1)(bt-b+1)(ct-c+a) \cdot$ $((a-1)(c-b)t-(b-1)(c-a))$

Note that the T_{15} is the translation of the section T_{13} by the torsion section T_4 .

6. FIBRATION 3

We go on to the next elliptic fibration. This corresponds to the elliptic divisor $N_3 + N_5 + 2T_0 + 2N_2 + 2T_2 + N_{12} + N_{24}$. It is therefore obtained by a 2-neighbor step from fibration 1.



We may take T_1 as the zero section, since its intersection with the class of the fiber is 1. This fibration has A_3, D_6 and six A_1 fibers, full 2-torsion, and Mordell-Weil rank 0. Therefore the sublattice of Néron-Severi lattice spanned by these sections and the trivial lattice has rank $2 + 3 + 6 + 6 \cdot 1 = 15$, and discriminant $4 \cdot 4 \cdot 2^6 / 2^2 = 64$, which matches that of $NS(X)$. Therefore it must be the full Néron-Severi lattice. The 2-torsion sections are T_3, T_4 and T_5 .

The elliptic parameter is given by

$$t_3 = -\frac{x_1}{a c t_1 (t_1 - b)} - \frac{4(a-1)(c-1)(b t_1 - a c)}{a c}.$$

A Weierstrass equation for this fibration is the following.

$$y^2 = \left(x + 4(a-1)b(c-b)t(t+4(b-a)(c-1)) \right) \cdot \\ \left(x - 4b(b-a)(c-1)t(t-4(a-1)(c-b)) \right) \cdot \\ \left(x - (t-4(a-1)(c-b))(t+4(b-a)(c-1)) \cdot \right. \\ \left. (a c t + 4(a-1)(b-a)(c-1)(c-b)) \right)$$

The reducible fibers are described in the following table.

Position	Reducible fiber	type
$t = \infty$	$N_3 + N_5 + 2T_0 + 2N_2 + 2T_2 + N_{12} + N_{24}$	D_6
$t = 0$	$N_{14} + T_{14} + N_{35} + T_{14}'''$	A_3
$t = 4(a-1)(c-b)$	$N_{15} + N_{15}''''$	A_1
$t = -4(c-1)(b-a)$	$N_{13} + N_{13}''''$	A_1
$t = -4(a-1)(c-1)(b-a)/a$	$N_{25}' + N_{25}''''$	A_1
$t = -4(a-1)(b-a)(c-b)/a$	$N_{45} + N_{45}''''$	A_1
$t = 4(a-1)(c-1)(c-b)/c$	$N_{23}' + N_{23}''''$	A_1
$t = -4(c-1)(b-a)(c-b)/c$	$N_{34} + N_{34}''''$	A_1

The (torsion) sections are indicated in the following table.

Torsion section	Equation
T_1	$x = y = \infty$.
T_3	$x = 4b(b-a)(c-1)t(t-4ac+4c+4ab-4b) \quad y = 0$
T_4	$x = (t-4(a-1)(c-b))(t+4(b-a)(c-1)) \cdot$ $(a c t + 4(a-1)(b-a)(c-1)(c-b))$ $y = 0$
T_5	$x = -4(a-1)b(c-b)t(t+4bc-4ac-4b+4a), \quad y = 0$

This fibration was studied by Keum [Ke1].

7. FIBRATION 4

This fibration corresponds to the elliptic divisor $2T_2 + N_{34}' + N_2 + N_{12} + N_{25}$, in the terminology of Section 4 (fibration 1). It is obtained by a 2-neighbor step from fibration 1.

We may take T_0 to be the zero section. This fibration has three D_4 fibers, an A_3 fiber, a 2-torsion section T_1 , and Mordell-Weil rank 0. The lattice spanned by the 2-torsion section and the trivial lattice has rank $2 + 3 \cdot 4 + 3 = 17$ and discriminant $4^3 \cdot 4/2^2 = 64$, so it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_4 = -(c+ab) + \frac{4(a-1)(b-1)(c-a)(c-b)t_1(t_1-ab)(t_1-c)}{x_1 + 4(a-1)(b-1)t_1(t_1-c)(c t_1 - ab)}.$$

A Weierstrass equation is given by

$$y^2 = x \left(x^2 + 4xt(t+c+ab)(t+ac+b)(t+bc+a) \right. \\ \left. + 16abc(t+c+ab)^2(t+ac+b)^2(t+bc+a)^2 \right)$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = -(c+ab)$	$2T_2 + N'_{34} + N_2 + N_{12} + N_{25}$	D_4
$t = -(b+ac)$	$2T_3 + N_3 + N_{13} + N_{35} + N'_{24}$	D_4
$t = -(a+bc)$	$2T_4 + N_4 + N_{14} + N_{45} + N'_{23}$	D_4
$t = \infty$	$N_5 + T_{15} + N_{15} + T_{15}^{(4)}$	A_3

The torsion section T_1 is given by $x = y = 0$.

8. FIBRATION 5

This fibration corresponds to the elliptic divisor $T_2 + N_{25} + T_{13} + N'_{34}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take N_{13} to be the zero section. This fibration has an A_7 fiber, two A_3 fibers, a 2-torsion section, and Mordell-Weil rank 2. We will describe explicit sections with canonical height pairing

$$\frac{1}{2} \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}.$$

The determinant of the height pairing matrix is 2. This implies that the sublattice of $\text{NS}(X)$ generated by these sections, along with the torsion section and the trivial lattice has rank 17 and discriminant $8 \cdot 4 \cdot 4 \cdot 2 / 2^2 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_5 = \frac{y_1 - 8(b-1)(b-a)(c-1)(c-a)t_1^2(t_1-a)(t_1-bc)}{2(t_1-ab)(t_1-c)(x_1+4(b-1)(c-1)t_1(t_1-a)(at_1-bc))} + \frac{(b-1)(c-a)t_1}{(t_1-ab)(t_1-c)}.$$

A Weierstrass equation for this elliptic fibration is given by

$$y^2 = x \left(x^2 + x \left(16(c-ab)^2 t^4 + 64(b-1)(c-a)(c+ab)t^3 \right. \right. \\ \left. \left. + 32(2abc^2 - 4bc^2 - ac^2 + 3c^2 - 4ab^2c + 2b^2c - a^2bc) \right. \right. \\ \left. \left. + 6abc - bc + 2a^2c - 4ac + 3a^2b^2 - ab^2 - 4a^2b + 2ab \right) t^2 \right. \\ \left. - 64(a-1)(b-1)(c-a)(c-b)t + 16(a-1)^2(c-b)^2 \right) \\ \left. + 4096a(b-1)b(b-a)(c-1)c(c-a)(t-1)^2 t^4 \right)$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_{13} + N'_{34} + T_2 + N_{25}$	A_3
$t = 0$	$T_1 + N_{15} + T_{15} + N_5 + T_0 + N_4 + T_4 + N_{14}$	A_7
$t = 1$	$T_3 + N'_{24} + T_{12} + N_{35}$	A_3

The two non-torsion sections mentioned above are N_{12} and N_{23} , and the torsion section is N_2 .

Section	Equation
N_{13}	$x = y = \infty$
N_2	$x = y = 0$
N_{12}	$x = 64(b-1)(b-a)(c-1)(c-a)(t-1)^2$ $y = 256(b-1)(b-a)(c-1)(c-a)(t-1)^2((c+ab)t^2 + 2(b-1)(c-a)t - 2bc + ac + c + ab + b - 2a)$
N_{23}	$x = 64b(b-a)(c-1)ct^2$ $y = 256b(b-a)(c-1)ct^2((c+ab-2a)t^2 + 2(b-1)(c-a)t - (a-1)(c-b))$

9. FIBRATION 6

This fibration corresponds to the elliptic divisor $T_5 + N'_{34} + T_2 + N_{25}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take N_{15} to be the zero section. This fibration has an A_7 fiber $T_1 + N_{14} + T_4 + N_4 + T_0 + N_3 + T_3 + N_{13}$, an A_3 fiber $T_5 + N'_{34} + T_2 + N_{25}$, two A_1 fibers $T_{12}^{(6)} + T_{12}$ and $T_{15} + T_{15}^{(6)}$, a 2-torsion section N_2 , and Mordell-Weil rank 3. The divisors N_{12} , N_{23} and N_{24} have the intersection pairing

$$\frac{1}{2} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The determinant of the height pairing matrix is 2. This implies that the sublattice of $\text{NS}(X)$ generated by these sections, along with the torsion section and the trivial lattice, has rank 17 and discriminant $8 \cdot 4 \cdot 2^2 \cdot 2/2^2 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_6 = \frac{y_1}{2(t_1 - c)(t_1 - ab)(x_1 + 4(a-1)(b-1)ct_1(t_1 - a)(t_1 - b))}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
 y^2 = x \bigg(& x^2 + x \left(16(c-ab)^2 t^4 + 32(2ab c^2 - b c^2 - a c^2 - a b^2 c + 2 b^2 c - a^2 b c \right. \\
 & \left. - b c + 2 a^2 c - a c - a b^2 - a^2 b + 2 a b) t^2 + 16(b-a)^2 (c-1)^2 \right) \\
 & \left. - 4096(a-1)a(b-1)bc(c-a)(c-b)(t-1)t^4(t+1) \right)
 \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_5 + N'_{34} + T_2 + N_{25}$	A_3
$t = 0$	$T_1 + N_{14} + T_4 + N_4 + T_0 + N_3 + T_3 + N_{13}$	A_7
$t = 1$	$T_{12}^{(6)} + T_{12}$	A_1
$t = -1$	$T_{15} + T_{15}^{(6)}$	A_1

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N_{15}	$x = y = \infty$
N_2	$x = y = 0$
N_{12}	$x = -64(a-1)(b-1)(c-a)(c-b)(t-1)(t+1)$ $y = -256(a-1)(b-1)(c-a)(c-b)(t-1)(t+1)(ct^2 + abt^2 + bc + ac - 2c - 2ab + b + a)$
N_{23}	$x = -64(a-1)bc(c-b)t^2$ $y = -256(a-1)bc(c-b)t^2((c+ab-2a)t^2 + (b-a)(c-1))$
N_{24}	$x = -64a(b-1)c(c-a)t^2$ $y = -256a(b-1)c(c-a)t^2((c+ab-2b)t^2 - (b-a)(c-1))$

10. FIBRATION 7

This fibration corresponds to the elliptic divisor $N_2 + T_2 + N_{12} + T_{12}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take T_0 to be the zero section. This fibration has four A_3 fibers indicated in the table below, a 2-torsion section T_1 , and Mordell-Weil rank 3. The sections N_{34} , N_{35} and N_{45} are all of height 1 and are orthogonal. Therefore the determinant of the height pairing matrix is 1. This implies that the sublattice of $\text{NS}(X)$ generated by these sections, along with the torsion section and the trivial lattice, has rank 17 and discriminant $4^4/2^2 \cdot 1 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_7 = \frac{2(a-1)(b-1)(c-1)t_1(x_1 - 4(t_1 - a)(t_1 - b)(t_1 - c)(t_1 - abc))}{y_1 + 2(t_1^2 - (a+b+c-1)t_1 + abc)x_1 + 8(a-1)(b-1)(c-1)t_1^2(t_1 - a)(t_1 - b)(t_1 - c)} - 1.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
y^2 = x \bigg(& x^2 + x \left((a^2 + b^2 + c^2 - 2ab - 2bc - 2ca - 2a - 2b - 2c + 1)t^4 \right. \\
& + 8(abc + ab + bc + ca)t^3 - 2((a+b+c+10)abc \\
& + (a+b)(b+c)(c+a) + ab + bc + ca)t^2 + 8abc(a+b+c+1)t \\
& + (a-1)^2b^2c^2 + (b-1)^2a^2c^2 + (c-1)^2a^2b^2 - 2abc(abc + a + b + c) \bigg) \\
& \left. + 16abc(t-1)^2(t-a)^2(t-b)^2(t-c)^2 \right)
\end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = 1$	$N_2 + T_2 + N_{12} + T_{12}$	A_3
$t = a$	$N_3 + T_3 + N_{13} + T_{13}$	A_3
$t = b$	$N_4 + T_4 + N_{14} + T_{14}$	A_3
$t = c$	$N_5 + T_5 + N_{15} + T_{15}$	A_3

Generators for the Mordell-Weil group are in the table below.

Section	Equation
T_0	$x = y = \infty$
T_1	$x = y = 0$
N_{34}	$x = 4ab(t-1)(t-a)(t-b)(t-c)$ $y = 4ab(t-1)(t-a)(t-b)(t-c) \cdot$ $((c-b-a+1)t^2 - 2(c-ab)t - abc + bc + ac - ab)$
N_{35}	$x = 4ac(t-1)(t-a)(t-b)(t-c)$ $y = 4ac(t-1)(t-a)(t-b)(t-c) \cdot$ $((c-b+a-1)t^2 - 2(ac-b)t + abc - bc + ac - ab)$
N_{45}	$x = 4bc(t-1)(t-a)(t-b)(t-c)$ $y = 4bc(t-1)(t-a)(t-b)(t-c) \cdot$ $((c+b-a-1)t^2 - 2(bc-a)t + abc + bc - ac - ab)$

11. FIBRATION 8

This fibration corresponds to the elliptic divisor $N_{13} + N_{15} + 2T_1 + 2N_{12} + 2T_2 + N_{25} + N'_{34}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take T_3 to be the zero section. This fibration has two D_6 fibers and two A_1 fibers and Mordell-Weil rank 1. The section T_{15} has height 1 and is a generator for the Mordell-Weil group, since the sublattice of $\text{NS}(X)$ it generates along with the trivial lattice has rank 17 and discriminant $4^2 \cdot 2^2 \cdot 1 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_8 = \frac{-abc(x_1 + 4(a-1)b(c-1)t_1(t_1-a)(t_1-c))}{4(b-a)(c-1)t_1^2(t_1-ab)(t_1-c)}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
 y^2 = & x^3 + x^2 t \left(2(b-a)(c-1)t^2 - (2abc^2 - bc^2 - 3a^2c^2 + 2ac^2 - ab^2c + 2b^2c \right. \\
 & + 2a^2bc - 6abc + 2bc + 2a^2c - ac + 2ab^2 - 3b^2 - a^2b + 2ab)t \\
 & \left. - 2abc(a-1)(c-b) \right) + xt^2(t-b)(t-ac) \left((b-a)^2(c-1)^2t^2 \right. \\
 & - (a-1)(b-a)(c-1)(c-b)(bc-3ac+c+ab-3b+a)t \\
 & \left. + abc(a-1)^2(c-b)^2 \right) + (a-1)^2(b-a)^2(c-1)^2(c-b)^2t^4(t-b)^2(t-ac)^2
 \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$N_{13} + N_{15} + 2T_1 + 2N_{12} + 2T_2 + N_{25} + N'_{34}$	D_6
$t = 0$	$N_3 + N_5 + 2T_0 + 2N_4 + 2T_4 + N_{45} + N'_{23}$	D_6
$t = b$	$N_{35} + N_{35}^{(8)}$	A_1
$t = ac$	$N'_{24} + N_{24}^{(8)}$	A_1

The section T_{15} has equation

$$x = 0, \quad y = (a-1)(b-a)(c-1)(c-b)t^2(t-b)(t-ac)$$

12. FIBRATION 8A

This is the elliptic fibration corresponding to the elliptic divisor $N_3 + N_{13} + 2T_{13} + 2N_{45} + 2T_5 + N_{35} + N'_{24}$. It may be obtained by a 2-neighbor step from fibration 2. But note that translation by the section $-T_{13}$ of fibration 2 transforms this elliptic divisor to $N_{12} + N_{13} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$, which is related to the elliptic divisor of fibration 8 by an automorphism in $\text{Aut}(D')$. Therefore this fibration is not new.

13. FIBRATION 9

This fibration corresponds to the elliptic divisor $N_{14} + N_{13} + 2T_1 + 2N_{12} + 2T_2 + N_{25} + N'_{34}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take T_4 to be the zero section. This fibration has a D_6 fiber, a D_5 fiber, and four A_1 fibers, as well as a 2-torsion section. These account for a sublattice of $\text{NS}(X)$ of rank 17 and discriminant $4^2 \cdot 2^4 / 2^2 = 64$, so they generate all of $\text{NS}(X)$. The Mordell-Weil rank of the elliptic fibration is zero.

An elliptic parameter is given by

$$t_9 = \frac{x_1 + 4(a-1)(b-1)t_1(t_1-c)(ct_1-ab)}{t_1^2(t_1-c)(t_1-ab)}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned} y^2 = x \bigg(& x^2 + 2xt \left(-abct^2 - 2(2abct^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - bc \right. \\ & + 2a^2c - ac - ab^2 - a^2b + 2ab)t - 16(a-1)(b-1)(c-a)(c-b) \bigg) \\ & + t^2(at + 4(a-1)(c-b))(bt + 4(b-1)(c-a)) \cdot \\ & \left. (act + 4(b-1)(c-a))(bct + 4(a-1)(c-b)) \right) \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$N_{14} + N_{13} + 2T_1 + 2T_{12} + 2T_2 + N_{25} + N'_{34}$	D_6
$t = 0$	$N_4 + N_3 + 2T_0 + 2N_5 + T_{15} + T_{15}^{(9)}$	D_5
$t = -4(a-1)(c-b)/a$	$N'_{23} + N_{23}^{(9)}$	A_1
$t = -4(b-1)(c-a)/b$	$N_{24}^{(9)} + N'_{24}$	A_1
$t = -4(b-1)(c-a)/(ac)$	$N_{35}^{(9)} + N_{35}$	A_1
$t = -4(a-1)(c-b)/(bc)$	$N_{45} + N_{45}^{(9)}$	A_1

The 2-torsion section T_3 is given by $x = y = 0$.

14. FIBRATION 10

This fibration corresponds to the elliptic divisor $T_1 + N_{15} + T_{15} + N_{24} + T_2 + N_{12}$, in the terminology of fibration 1. It is obtained by a 2-neighbor step from fibration 1.

We may take N_{14} to be the zero section. This fibration has two A_5 fibers and two A_1 fibers, a 2-torsion section T_{13} , and Mordell-Weil rank 3. The divisors N_2 , N_5 and N'_{34} have the intersection pairing

$$\frac{1}{3} \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

The determinant of the height pairing matrix is $16/9$. Therefore the sublattice of $\text{NS}(X)$ generated by these sections, along with the torsion section and the trivial lattice, has rank 17 and discriminant $(6^2 \cdot 2^2/2^2) \cdot 16/9 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{10} = \frac{y_1 - 8(a-1)(b-1)(a-c)(b-c)t_1^2(t_1-c)(t_1-ab)}{(x_1 + 4(a-1)(b-1)t_1(t_1-c)(ct_1-ab))t_1(t_1-b)} - \frac{2(a-1)(c-b)}{t_1-b}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned} y^2 = x \bigg(& x^2 + x \left(b^2 t^4 - 8(a-1)b(c-b)t^3 - 8(4ab c^2 - 2b c^2 - 2a c^2 - 2a b^2 c \right. \\ & + b^2 c - 2a^2 b c - 3a b c + 4b c + 4a^2 c - 2a c + 4a b^2 - 3b^2 - 2a^2 b \\ & + a b) t^2 + 32(a-1)(b-a)(c-1)(c-b)t + 16(b-a)^2(c-1)^2 \bigg) \\ & \left. + 128(a-1)a(b-1)c(c-a)(c-b)t^3(t+2c-2)(bt-2b+2a) \right) \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_1 + N_{15} + T_{15} + N_{24} + T_2 + N_{12}$	A_5
$t = 0$	$T_{14} + N_4 + T_0 + N_3 + T_3 + N_{35}$	A_5
$t = -2(c-1)$	$T_{25} + N'_{23}$	A_1
$t = 2(b-a)/b$	$T_{45}^{(10)} + N_{45}$	A_1

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N_{14}	$x = y = \infty$
T_{13}	$x = y = 0$
N_2	$x = 16(a-1)ac(c-b)t^2$ $y = 16(a-1)ac(c-b)t^2(bt^2 + 4(b-a)(c-1)(t-1))$
N_5	$x = 16a(b-1)c(c-a)t^2$ $y = -16a(b-1)c(c-a)t^2(bt^2 + 4(b-a)(c-1))$
N'_{34}	$x = 16(a-1)(b-1)c(c-a)t^2$ $y = -16(a-1)(b-1)c(c-a)t^2(bt^2 + 4(c-b)t - 4(b-a)(c-1))$

15. FIBRATION 11

This corresponds to the elliptic divisor $N_5 + N_3 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$. It is obtained by a 2-neighbor step from fibration 1.

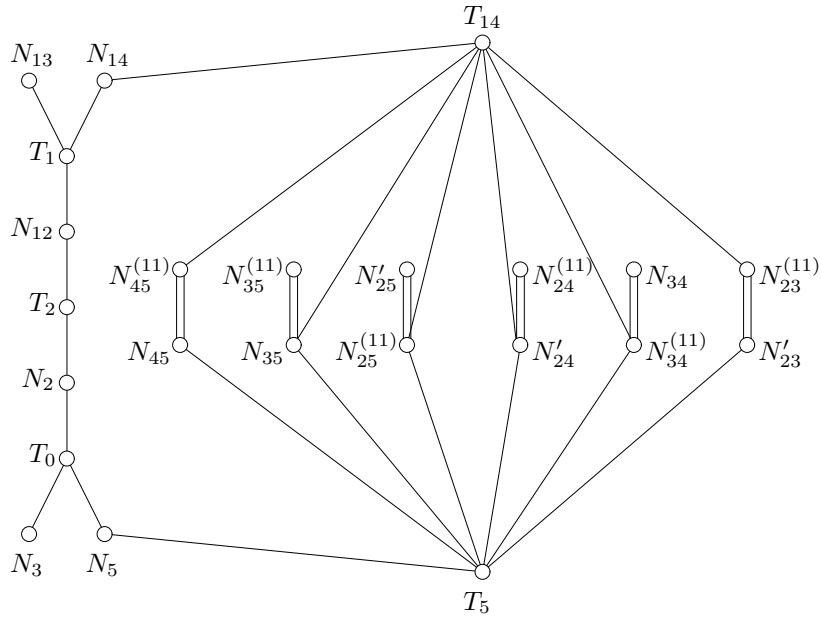
We may take T_5 as the zero section. This fibration has a D_8 fiber, six A_1 fibers, a 2-torsion section T_4 , and Mordell-Weil rank 1. A non-torsion section is given by T_{14} ; it has height 1 and so the sublattice of $\text{NS}(X)$ it generates with the torsion section and the trivial lattice has rank 17 and discriminant $4 \cdot 2^6/2^2 \cdot 1 = 64$. It is therefore all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{11} = \frac{x_1 + 4(a-1)a(b-1)bc t_1}{4(a-1)(c-b)t_1^2} + \frac{b(c-1)t_1 - ac^2 - b^2c + bc + ac}{c-b}.$$

A Weierstrass equation for this fibration is the following.

$$\begin{aligned} y^2 = x \bigg(& x^2 + x \left(4(a-1)(c-b)t^3 - 4(2abc^2 - bc^2 - 3a^2c^2 + 2ac^2 - ab^2c \right. \\ & + 2b^2c + 2a^2bc - 3abc - bc + 2a^2c - ac - ab^2 - a^2b + 2ab)t^2 \\ & + 4(ab^2c^3 - 4a^2bc^3 + abc^3 + 3a^3c^3 - a^2c^3 + b^3c^2 + a^2b^2c^2 \\ & - 4ab^2c^2 - b^2c^2 - a^3bc^2 + 6a^2bc^2 + abc^2 - 4a^3c^2 + a^2c^2 - ab^3c \\ & + a^2b^2c + 3ab^2c + a^3bc - 4a^2bc - abc + a^3c - a^2b^2 + a^2b)t \\ & + 4a(b-a)(c-1)c(ab^2c^2 - a^2c^2 - b^2c - abc + bc + a^2c + ab^2 - ab) \\ & \left. + 16a(b-1)b(b-a)(c-1)c(c-a)t(t+ac-b) \right. \\ & \left. (t-bc+ac+ab-a)(t-bc+ac+c-a) \right) \end{aligned}$$



The reducible fibers are described in the following table.

Position	Reducible fiber	type
$t = \infty$	$N_5 + N_3 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1$ $+ N_{13} + N_{14}$	D_8
$t = 0$	$N_{35} + N_{35}^{(11)}$	A_1
$t = bc - ac - c + a$	$N_{34}^{(11)} + N_{34}$	A_1
$t = (b-a)c$	$N_{45} + N_{45}^{(11)}$	A_1
$t = bc - ac - ab + a$	$N_{25}^{(11)} + N_{25}'$	A_1
$t = -a(c-1)$	$N_{23}' + N_{23}^{(11)}$	A_1
$t = b - ac$	$N_{24}' + N_{24}^{(11)}$	A_1

Generators for the Mordell-Weil group are in the table below.

Torsion section	Equation
T_5	$x = y = \infty$
T_4	$x = y = 0$
T_{14}	$x = 4a(b-1)b(b-a)(c-1)c(c-a)$ $y = -8a(b-1)b(b-a)(c-1)c(c-a)(t+ac-a) \cdot (t-bc+ac)$

We note for future reference that the sum of the sections T_4 and T_{14} in this fibration gives the sections T_{15} .

16. FIBRATION 11A

This fibration corresponds to the divisor $N_2 + N_4 + 2T_0 + 2N_3 + 2T_{13} + 2N_{45} + 2T_5 + N_{15} + N_{35}$. It may be obtained by a 2-neighbor step from fibration 2. First note that translation by the section $-T_{15}$ of fibration 2 transforms this divisor to $N_2 + N_3 + 2T_0 + 2N_4 + 2T_4 + 2N_{45} + 2T_5 + N_{15} + N_{35}$, which is related to the elliptic divisor of fibration 11 by an element of $\text{Aut}(D')$. Hence this fibration is not new.

17. FIBRATION 12

This corresponds to the elliptic divisor $T_2 + 2N_2 + 3T_0 + 2N_3 + T_4 + 2N_4 + T_{13}$ in fibration 2. It is obtained from a 2-neighbor step from fibration 2. The automorphism which is translation by the section T_4 in fibration 2 transforms it to the elliptic divisor $T_0 + 2N_2 + 3T_2 + 2N_{12} + T_1 + 2N_{23} + T_{15}$, which is the one that we shall work with.

We may take N_4 as the zero section. This fibration has an E_6 and a D_5 fiber, and Mordell-Weil rank 4.

The divisors N_{13} , N''_{14} , N''_{34} and N''_{25} have the intersection pairing

$$\frac{1}{3} \begin{pmatrix} 5 & 1 & -1 & -1 \\ 1 & 5 & 1 & 1 \\ -1 & 1 & 5 & -1 \\ -1 & 1 & -1 & 5 \end{pmatrix}.$$

The determinant of the height pairing matrix is $16/3$. Therefore the sublattice of $\text{NS}(X)$ generated by these sections, along with the trivial lattice has rank 17 and discriminant $3 \cdot 4 \cdot 16/3 = 64$. So it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{12} = \frac{y_2 + 2(b-1)(c-a)(t_2-1)x_2}{(b-a)(c-1)(x_2 + 4(bt_2 - b + 1)(ct_2 - c + a)(act_2 - ct_2 - abt_2 + bt_2 - bc + c + ab - a))}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
y^2 = & x^3 + x^2 t \left((ab^2c^2 - 2bc^2 + ac^2 - 2ab^2c + b^2c + a^2bc + 3abc + bc \right. \\
& \left. - 2a^2c - 2ac + ab^2 - 2a^2b - 2ab + 3a^2) t - 4a(b-1)(c-a) \right) \\
& - x t^2 \left((a-1)(b-1)b(b-a)(c-1)c(c-a)(c-b)t^3 + 2(ab^2c^4 - b^2c^4 \right. \\
& - a^2bc^4 + abc^4 + a^2b^3c^3 - 3ab^3c^3 + b^3c^3 - a^3b^2c^3 + a^2b^2c^3 + 3ab^2c^3 \\
& + b^2c^3 + a^3bc^3 - 3a^2bc^3 - 3abc^3 + a^3c^3 + a^2c^3 - a^2b^4c^2 + ab^4c^2 \\
& + a^3b^3c^2 + 3a^2b^3c^2 + ab^3c^2 - b^3c^2 - 3a^3b^2c^2 - 9a^2b^2c^2 - 3ab^2c^2 \\
& + a^4bc^2 + 5a^3bc^2 + 11a^2bc^2 + abc^2 - a^4c^2 - 5a^3c^2 - a^2c^2 + a^2b^4c \\
& - ab^4c - 3a^3b^3c - 3a^2b^3c + ab^3c + a^4b^2c + 11a^3b^2c + 5a^2b^2c + ab^2c \\
& - 5a^4bc - 11a^3bc - 5a^2bc + 4a^4c + 4a^3c + a^3b^3 + a^2b^3 - a^4b^2 \\
& - 5a^3b^2 - a^2b^2 + 4a^4b + 4a^3b - 3a^4) t^2 + 4a(b-1)(c-a)(abc^2 \\
& - 2bc^2 + ac^2 - 2ab^2c + b^2c + a^2bc + 4abc + bc - 3a^2c - 2ac + ab^2 \\
& - 2a^2b - 3ab + 4a^2) t - 8a^2(b-1)^2(c-a)^2 \Big) / 2 \\
& + t^4 \left((a-1)b(b-a)(c-1)c(c-b)t^2 - 4a(b-1)(b-a)(c-1)(c-a)t \right. \\
& \left. - 4a(b-1)(c-a)(ac+b-2a) \right)^2 / 16
\end{aligned}$$

This fibration has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_0 + 2N_2 + 3T_2 + 2N_{12} + T_1 + 2N_{23} + T_{15}$	E_6
$t = 0$	$N'_{24} + N_{35} + 2T_5 + 2N_{45} + T_4 + T_{13}$	D_5

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N_4	$x = y = \infty$
N_{13}	$x = (b-a)(c-1)(bc-a)t^2$ $y = -t^2((a-1)b(b-a)(c-1)c(c-b)t^2$ $-4(b-1)b(b-a)(c-1)c(c-a)t$ $+4a(b-1)(c-a)(2bc-ac-b))/4$
N''_{14}	$x = 0$ $y = t^2((a-1)b(b-a)(c-1)c(c-b)t^2$ $-4a(b-1)(b-a)(c-1)(c-a)t$ $-4a(b-1)(c-a)(ac+b-2a))/4$
N''_{34}	$x = -a(b-1)(ac-ab+b-a)t^2$ $y = -t^2((a-1)b(b-a)(c-1)c(c-b)t^2$ $+4(a-1)a(b-1)b(c-a)(c-b)t$ $+4a(b-1)(c-a)(ac-2ab+b))/4$
N''_{25}	$x = -(c-a)(ac-c+b-a)t^2$ $y = -t^2((a-1)b(b-a)(c-1)c(c-b)t^2$ $+4(a-1)(b-1)c(c-a)(c-b)t$ $+4a(b-1)(c-a)(ac-2c+b))/4$

18. FIBRATION 13

This fibration corresponds to the elliptic divisor $T_4 + T_1 + 2N_{14} + T_{14} + T_{14}'''$, in the terminology of fibration 3. It is obtained from fibration 3 by a 2-neighbor step. We may take N_{24} to be the zero section. This elliptic fibration has an E_6 fiber, a D_4 fiber, and Mordell-Weil rank 5. The sections $N_{13}, N_{15}, N_{34}''', N_{45}''', N_{23}'''$ have intersection pairing

$$\frac{1}{3} \begin{pmatrix} 5 & 1 & 1 & -1 & -1 \\ 1 & 5 & -1 & 1 & 1 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 1 & 1 & 5 & -1 \\ -1 & 1 & 1 & -1 & 5 \end{pmatrix}.$$

The determinant of the intersection matrix is $16/3$. Therefore the sublattice of $\text{NS}(X)$ generated by these sections and the trivial lattice has rank 17 and discriminant $3 \cdot 4 \cdot (16/3) = 64$.

An elliptic parameter is given by

$$t_{13} = \frac{(a-1)(b-a)(c-1)(c-b)y_3}{t_3^2 \left(x_3 - (t_3 - 4(a-1)(c-b))(t_3 + 4(b-a)(c-1))(act_3 + 4(a-1)(b-a)(c-1)(c-b)) \right)}.$$

The Weierstrass equation for this elliptic fibration turns out to be invariant under all the permutations of the Weierstrass points of the genus 2 curve C . It may be written in terms of the Igusa-Clebsch invariants of the curve as

$$y^2 = x^3 - 108xt^4(48t^2 + I_4) + 108t^4(72I_2t^4 + (4I_4I_2 - 12I_6)t^2 + 27I_{10}).$$

The reducible fibers are as follows.

Position	Reducible fiber	type
$t = \infty$	$T_4 + T_1 + 2N_{14} + T_{14} + T_{14}'''$	D_4
$t = 0$	$T_2 + 2N_2 + 3T_0 + 2N_3 + T_3 + 2N_5 + T_5$	E_6

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N_{24}	$x = y = \infty$
N_{13}	$x = 12 (a b c^2 + b c^2 - 2 a c^2 + a b^2 c - 2 b^2 c - 2 a^2 b c + b c + a^2 c$ $+ a c + a b^2 + a^2 b - 2 a b) t^2$ $y = 54 t^2 (4 (b c - a c - c - a b + b + a) t^2 + I_5)$
N_{15}	$x = -12 (2 a b c^2 - b c^2 - a c^2 - a b^2 c - b^2 c - a^2 b c + 2 b c + 2 a^2 c$ $- a c + 2 a b^2 - a^2 b - a b) t^2$ $y = -54 t^2 (4 (b c + a c - c - a b - b + a) t^2 + I_5)$
N_{34}'''	$x = 12 (a b c^2 + b c^2 - 2 a c^2 - 2 a b^2 c + b^2 c + a^2 b c - 2 b c$ $+ a^2 c + a c + a b^2 - 2 a^2 b + a b) t^2$ $y = -54 t^2 (-4 (b c - a c - c + a b - b + a) t^2 + I_5)$
N_{45}'''	$x = 12 (a b c^2 - 2 b c^2 + a c^2 - 2 a b^2 c + b^2 c + a^2 b c + b c$ $- 2 a^2 c + a c + a b^2 + a^2 b - 2 a b) t^2$ $y = 54 t^2 (4 (b c - a c + c + a b - b - a) t^2 + I_5)$
N_{23}'''	$x = -12 (2 a b c^2 - b c^2 - a c^2 - a b^2 c + 2 b^2 c - a^2 b c - b c$ $- a^2 c + 2 a c - a b^2 + 2 a^2 b - a b) t^2$ $y = 54 t^2 (-4 (b c + a c - c - a b + b - a) t^2 + I_5)$

with $I_5 = a b c (a - 1) (b - 1) (c - 1) (a - b) (b - c) (c - a)$ being a square root of I_{10} .

19. FIBRATION 14

This fibration comes from the elliptic divisor $T_1 + N_{13} + T_3 + N_{23} + T_2 + N_{12}$. It is obtained from fibration 1 by a 2-neighbor step.

We may take N_{15} to be the zero section. This fibration has two A_5 fibers and Mordell-Weil rank 5. The divisors N_2, N_3, N_{24}, N_{34} and T_{12} have the intersection pairing

$$\frac{1}{3} \begin{pmatrix} 4 & 0 & 0 & 2 & 0 \\ 0 & 4 & 2 & 0 & 4 \\ 0 & 2 & 4 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 0 & 4 & 2 & 0 & 7 \end{pmatrix}.$$

The determinant of the height pairing matrix is $16/9$. Therefore the sublattice of $\text{NS}(X)$ generated by these sections, along with the trivial lattice, has rank 17 and discriminant $6^2 \cdot 16/9 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{14} = \frac{y_1}{2(a-1)(c-b)t_1^2(t_1-a)(x_1+4a(b-1)(c-1)t_1(t_1-b)(t_1-c))}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
 y^2 = & x^3 + x^2 \left((a-1)^2 a^2 (c-b)^2 t^4 + 2(2ab c^2 + 2b c^2 - 4a c^2 + 2a b^2 c + 2b^2 c \right. \\
 & \left. - 4a^2 b c - 6a b c - 4b c + 5a^2 c + 5a c - 4a b^2 + 5a^2 b + 5a b - 6a^2) t^2 + 1 \right) \\
 & - 8(b-1)(b-a)(c-1)(c-a) t^2 x \left((a-1)^2 a^2 (c-b)^2 t^4 - 2(a c^2 + a^2 b c \right. \\
 & \left. + a b c + b c - 2a^2 c - 2a c + a b^2 - 2a^2 b - 2a b + 3a^2) t^2 + 1 \right) \\
 & + 16(b-1)^2 (b-a)^2 (c-1)^2 (c-a)^2 t^4 \left((a-1)^2 a^2 (c-b)^2 t^4 + 2a(a c + c \right. \\
 & \left. + a b + b - 2a) t^2 + 1 \right)
 \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_1 + N_{13} + T_3 + N_{23} + T_2 + N_{12}$	A_5
$t = 0$	$T_5 + N_5 + T_0 + N_4 + T_4 + N_{45}$	A_5

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N_{14}	$x = y = \infty$
N_2	$x = -4(b-1)(c-a)(ac+b-a)t^2$ $y = -4(b-1)bc(c-a)t^2(a^2ct^2 - act^2 - a^2bt^2 + abt^2 - 1)$
N_3	$x = -4(b-a)(c-1)(c+ab-a)t^2$ $y = 4b(b-a)(c-1)ct^2(a^2ct^2 - act^2 - a^2bt^2 + abt^2 + 1)$
N_{24}	$x = -4(b-1)(b-a)(c-a)t^2$ $y = -4(b-1)(b-a)c(c-a)t^2(a^2ct^2 - act^2 - a^2bt^2 + abt^2 + 1)$
N_{34}	$x = -4a(b-1)(b-a)(c-1)t^2$ $y = 4(b-1)(b-a)(c-1)ct^2(a^2ct^2 - act^2 - a^2bt^2 + abt^2 - 1)$
T_{12}	$x = -4(b-1)(c-a)t(a^2ct^2 - act^2 - a^2bt^2 + abt^2 + ct + abt - 2at + 1)$ $y = -4(b-1)(c-a)t \left((a-1)^2 a^2 (c-b)^2 t^4 \right. \\ \left. - (a-1)a(c-b)(bc+ac-3c-3ab+b+3a)t^3 \right. \\ \left. - 2(ac^2 - c^2 + a^2bc - 3abc + bc - 2a^2c + 3ac - a^2b^2 + ab^2 \right. \\ \left. + 3a^2b - 2ab - a^2) t^2 - (bc+ac-3c-3ab+b+3a)t + 1 \right)$

20. FIBRATION 15

This fibration corresponds to the elliptic divisor $N_2 + N_3 + 2T_0 + 2N_4 + 2T_4 + 2N_{45} + 2T_5 + N_{35} + N'_{24}$, in the notation of fibration 2. Under the automorphism which is translation by T_4 in fibration 2, this divisor is taken to the elliptic divisor $N_2 + N_{23} + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$, which we shall work with.

We may take T_0 to be the zero section. This fibration is obtained from fibration 2 by a 2-neighbor step. It has D_8 , D_4 and A_3 fibers. The trivial lattice has rank 17 and discriminant $4 \cdot 4 \cdot 4 = 64$, and must therefore be all of $\text{NS}(X)$. The Mordell-Weil rank is 0.

An elliptic parameter is given by

$$t_{15} = \frac{x_2}{4a(b-a)(c-1)t_2^2}.$$

A Weierstrass equation for this fibration is given by

$$\begin{aligned}
y^2 = & x^3 + x^2 t \left(a(b-a)(c-1)t^2 - (2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c \right. \\
& - a^2bc - bc - a^2c + 2ac - ab^2 + 2a^2b - ab)t + b^2c^2 + 2abc^2 \\
& \left. - 3bc^2 - ac^2 + c^2 - 3ab^2c + 2b^2c - a^2bc + 3abc - bc + a^2b^2 - ab^2 \right) \\
& + x(a-1)(b-1)bc(c-a)(c-b)(t-1)t^2(act + bt - 2at - 2bc \\
& - ac + 2c + 2ab - b) + (a-1)^2(b-1)^2b^2c^2(c-a)^2(c-b)^2(t-1)^2t^3
\end{aligned}$$

It has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$N_2 + N_{23} + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$	D_8
$t = 0$	$N_4 + 2T_4 + N_{34} + N''_{13} + N'_{25}$	D_4
$t = 1$	$N_3 + T_{13} + N''_{14} + T_{13}^{(15)}$	A_3

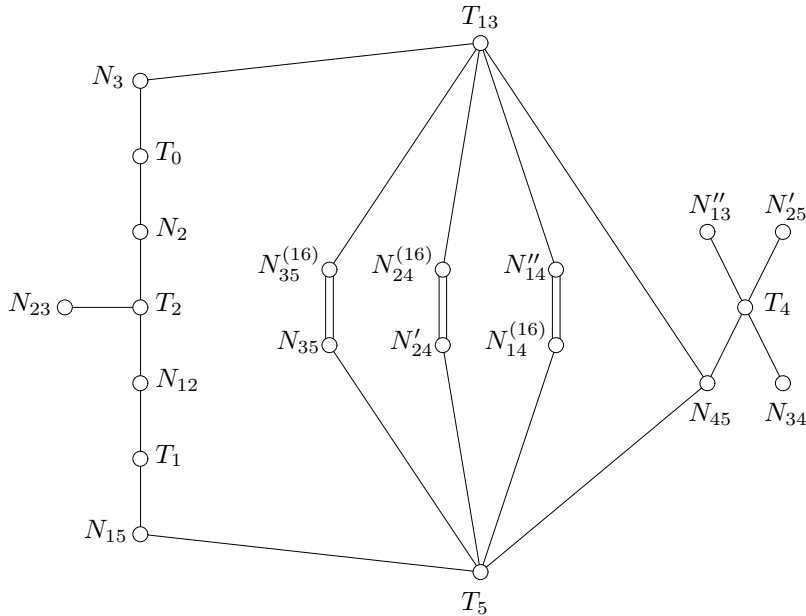
21. FIBRATION 15A

This fibration corresponds to the elliptic divisor $N_2 + N_4 + 2T_0 + 2N_3 + 2T_{13} + 2N_{45} + 2T_5 + N_{35} + N'_{24}$. It is obtained from fibration 2 by a 2-neighbor step. However, translation by $-T_{13}$ in fibration 2 turns this divisor into $N_2 + N_{23} + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$, which is the elliptic divisor of fibration 15. Therefore this fibration is not new.

22. FIBRATION 16

This elliptic fibration comes from the divisor $N_{23} + 2T_2 + 3N_2 + 4T_0 + 2N_3 + 3N_4 + 2T_4 + N_{45}$ in the terminology of fibration 2. Translation by T_4 in that fibration takes this to the elliptic divisor $N_{15} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_3$ that we shall work with.

We may take T_5 to be the zero section. This fibration has an E_7 fiber, a D_4 fiber, and three A_1 fibers. The Mordell-Weil group has rank 1, and is generated by a section T_{13} of height $4 - 3/2 - 1/2 - 1/2 - 1/2 = 1$. The sublattice of $\text{NS}(X)$ generated by this section, along with the trivial lattice, has rank 17 and discriminant $2 \cdot 4 \cdot 2^3 \cdot 1 = 64$ and must be all of $\text{NS}(X)$.



An elliptic parameter is given by

$$t_{16} = \frac{-x_2}{4a(b-1)(b-a)(c-1)(c-a)t_2}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned} y^2 = & x^3 - 4x^2t \left((2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - bc \right. \\ & \left. - a^2c + 2ac - ab^2 + 2a^2b - ab)t - (2bc + ac - 2c - 2ab + b) \right) \\ & - 16xt^2(bt - at - 1)(act - at - 1) \left((a-1)(b-1)bc(c-a)(c-b)t \right. \\ & \left. - (b^2c^2 + 2abc^2 - 3bc^2 - ac^2 + c^2 - 3ab^2c + 2b^2c - a^2bc + 3abc \right. \\ & \left. - bc + a^2b^2 - ab^2) \right) \\ & + 64(a-1)(b-1)bc(c-a)(c-b)t^3(bt - at - 1)^2(act - at - 1)^2 \end{aligned}$$

The equation of the section T_{13} is

$$\begin{aligned} x &= 4(bt - at - 1)(act - at - 1) \\ y &= 8(bt - at - 1)(act - at - 1)(bct - ct - abt + at + 1). \end{aligned}$$

The reducible fibers are as follows:

Position	Reducible fiber	type
$t = \infty$	$N_{15} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_3$	E_7
$t = 0$	$N_{45} + 2T_4 + N_{34} + N'_{25} + N''_{13}$	D_4
$t = \frac{1}{b-a}$	$N'_{24} + N_{24}^{(16)}$	A_1
$t = \frac{1}{a(c-1)}$	$N_{35} + N_{35}^{(16)}$	A_1
$t = \frac{-1}{(b-1)(c-a)}$	$N_{14}^{(16)} + N''_{14}$	A_1

23. FIBRATION 16A

This elliptic fibration arises from the elliptic divisor $N_{23} + 2T_2 + 3N_2 + 4T_0 + 2N_4 + 3N_3 + 2T_{13} + N_{13}$. It is obtained from fibration 2 by a 2-neighbor step. But translation by $-T_{13}$ transforms this divisor to $N_{13} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_4$, which is in the orbit of the elliptic divisor of fibration 16 by $\text{Aut}(D')$. Therefore this fibration is not new.

24. FIBRATION 17

This fibration corresponds to the elliptic divisor $2T_4 + N_{14} + N''_{13} + N_{34} + N'_{25}$, in the notation of fibration 2. Under the automorphism which is translation by T_4 in that fibration, this divisor is taken to the elliptic divisor $2T_1 + N_{14} + N_{13} + N''_{34} + N''_{25}$, which we shall work with.

We may take T_4 to be the zero section. This fibration is obtained from fibration 2 by a 2-neighbor step. It has a D_7 and two D_4 fibers. The trivial lattice has rank 17 and discriminant $4 \cdot 4 \cdot 4 = 64$, and must therefore be all of $\text{NS}(X)$. The Mordell-Weil rank is 0.

An elliptic parameter is given by

$$t_{17} = -\frac{x_2}{4(t_2 - 1)(bt_2 - b + 1)(ct_2 - c + a)((a-1)(c-b)t_2 - (b-1)(c-a))} - \frac{1}{t_2 - 1}.$$

A Weierstrass equation for this fibration is given by

$$\begin{aligned}
y^2 = & x^3 - x^2(t-1)t \left((a-1)bc(c-b)t + (bc^2 - ac^2 + ab^2c - a^2bc \right. \\
& \left. - 3abc - bc + 2a^2c + 2ac - ab^2 + 2a^2b + 2ab - 3a^2) \right) \\
& + a(b-1)(c-1)(b-a)(c-a)(bc - ac - c - ab - b + 3a)(t-1)^2 t^2 x \\
& + a^2(b-1)^2(b-a)^2(c-1)^2(c-a)^2(t-1)^3 t^3
\end{aligned}$$

It has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$N_4 + N_3 + 2T_0 + 2N_2 + 2T_2 + N_{23} + T_{15} + T_{15}^{(17)}$	D_7
$t = 0$	$N_{14} + 2T_1 + N_{13} + N_{34}'' + N_{25}''$	D_4
$t = 1$	$N_{45} + 2T_5 + N_{35} + N_{24}' + N_{45}^{(17)}$	D_4

25. FIBRATION 18

This fibration corresponds to the elliptic divisor $N_{23} + 2T_2 + 3N_2 + 4T_0 + 2N_4 + 3N_3 + 2T_{13} + N_{45}$, in the terminology of section 2. The automorphism which is the inverse of translation by T_{13} in that fibration transforms this divisor to $N_{15} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_4$, which is the divisor we shall work with.

We may take T_5 to be the zero section. This fibration is obtained from fibration 2 by a 2-neighbor step. It has an E_7 fiber, an A_3 fiber and five A_1 fibers, as well as a 2-torsion section. The trivial lattice and the torsion section span a sublattice of $\text{NS}(X)$ of rank 17 and discriminant $2 \cdot 4 \cdot 2^5 / 2^2 = 64$. Hence this sublattice is all of $\text{NS}(X)$. The Mordell-Weil rank of this fibration is 0.

An elliptic parameter is given by

$$t_{18} = -\frac{x_2}{4a(b-1)(b-a)(c-1)(c-a)t_2} + \frac{t_2}{(b-1)(c-a)}.$$

A Weierstrass equation for this fibration is given by

$$\begin{aligned}
y^2 = & x \left(x^2 + x \left(-4(2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - bc - a^2c + 2ac \right. \right. \\
& \left. \left. - ab^2 + 2a^2b - ab) t^2 + 8(bc + ac - c - ab + b - a)t - 8 \right) - 16(bt - at - 1) \cdot \right. \\
& \left. (act - at - 1)(act - ct - abt + bt - 1)(bct - abt - 1)(bct - ct - 1) \right)
\end{aligned}$$

It has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$N_{15} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_4$	E_7
$t = 0$	$N_{45} + T_{13} + N_{14}'' + T_{13}^{(18)}$	A_3
$t = \frac{1}{b-a}$	$N_{24}' + N_{24}^{(18)}$	A_1
$t = \frac{1}{a(c-1)}$	$N_{35} + N_{35}^{(18)}$	A_1
$t = \frac{1}{c(b-1)}$	$N_{34}^{(18)} + N_{34}$	A_1
$t = \frac{1}{(a-1)(c-b)}$	$N_{13}^{(18)} + N_{13}''$	A_1
$t = \frac{1}{b(c-a)}$	$N_{25}^{(18)} + N_{25}'$	A_1

The 2-torsion section T_4 is $x = y = 0$.

26. FIBRATION 18A

This fibration corresponds to the elliptic divisor $N_{12} + 2T_2 + 3N_2 + 4T_0 + 2N_3 + 3N_4 + 2T_4 + N_{45}$. It is, however, in the orbit of the (modified) elliptic divisor $N_{15} + 2T_1 + 3N_{12} + 4T_2 + 2N_{23} + 3N_2 + 2T_0 + N_4$ of fibration 18 under $\text{Aut}(D')$, so this fibration is not new.

27. FIBRATION 19

This fibration comes from the elliptic divisor $T_{12} + T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$. It is obtained from fibration 1 by a 2-neighbor step.

We may take N'_{23} to be the zero section. This fibration has two D_5 fibers and Mordell-Weil rank 5. The divisors N_{24} , N_{25} , T_3 , T_{14} and N'_{45} have the intersection pairing

$$\frac{1}{2} \begin{pmatrix} 4 & 2 & 2 & 2 & 0 \\ 2 & 6 & 3 & 1 & 2 \\ 2 & 3 & 3 & 1 & 0 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 2 & 0 & 0 & 4 \end{pmatrix}.$$

The determinant of the height pairing matrix is 4. Therefore the sublattice of $\text{NS}(X)$ generated by these sections, along with the trivial lattice, has rank 17 and discriminant $4^2 \cdot 4 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{19} = \frac{y_1 + 8(t_1 - a)(t_1 - b)(t_1 - c)(t_1 - ab)(t_1 - bc)(t_1 - ac)}{(a-1)(b-1)(c-a)(c-b)t_1^2(x_1 - 4(t_1 - a)(t_1 - b)(t_1 - c)(t_1 - abc))} + \frac{2(t_1 - ab)(t_1 - c)}{(a-1)(b-1)(c-a)(c-b)t_1^2}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned} y^2 = & x^3 + x^2 t \left(-18(a-1)a(b-1)bc(c-a)(c-b)t^2 + 36(3b^2c^2 - 2ab^2c^2 \right. \\ & - 2bc^2 + ac^2 - 2ab^2c - 2b^2c + a^2bc + 3abc + bc + a^2c - 2ac + ab^2 \\ & \left. - 2a^2b + ab)t - 72 \right) \\ & - 648(b-a)(c-1)xt^3 \left((a-1)a(b-1)bc(c-a)(c-b)(2bc - c - ab)t^2 \right. \\ & - 2(3b^3c^3 - ab^2c^3 - 4b^2c^3 + abc^3 + bc^3 - 4ab^3c^2 - b^3c^2 + a^2b^2c^2 \\ & + 6ab^2c^2 + b^2c^2 + a^2bc^2 - 4abc^2 - a^2c^2 + ac^2 + a^2b^3c + ab^3c - 4a^2b^2c \\ & \left. + ab^2c - a^3bc + 3a^2bc - abc + a^3b^2 - a^2b^2)t + 4(2bc - c - ab) \right) \\ & + 2916(b-a)^2(c-1)^2t^4 \left((a-1)a(b-1)bc(c-a)(c-b)t^2 \right. \\ & \left. - 4(b-1)bc(c-a)t + 4 \right)^2 \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_{12} + T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$	D_5
$t = 0$	$T_5 + T_{15} + 2N_5 + 2T_0 + N_3 + N_4$	D_5

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N'_{23}	$x = y = \infty$
N_{24}	$x = 18t((a-1)a(b-1)bc(c-a)(c-b)t^2 - 2(b^2c^2 - abc^2 - bc^2 - ab^2c + 3abc + a^2c - ac - a^2b)t + 4)$ $y = -54(c-1)t^2((a-1)a(b-1)b(b+a)c(c-a)(c-b)t^2 + 4ab(c^2 + abc - 2bc - 2ac + c + ab)t + 4(b+a))$
N_{25}	$x = 18((b-1)c(c-a)t - 2)((a-1)ab(c-a)^2(c-b)t^2 - 2(c-a)(bc - 2ab + a)t - 4)/(c-a)^2$ $y = -54((a-1)a(b-1)bc(c-a)^4(c-b)(bc - ac + 2ab - b - a)t^4 - 4a(b-1)(c-a)^3(2abc^2 - bc^2 - ac^2 - 3ab^2c + 2b^2c + a^2bc + abc - bc - a^2b^2 + ab^2)t^3 + 4(c-a)^2(2abc^2 - bc^2 - ac^2 - 4ab^2c + 2b^2c + 2a^2bc + 5abc - bc - a^2c - 3ac - 6a^2b^2 + 2ab^2 + 6a^2b - ab - a^2)t^2 - 16(c-a)(c - 3ab + 2a)t - 32)/(c-a)^3$
T_3	$x = -36(b-a)(c-1)(bc-a)t^2$ $y = 54(b-a)(c-1)t^2((a-1)a(b-1)bc(c-a)(c-b)t^2 - 4a(b-1)(c-a)t + 4)$
T_{14}	$x = 0$ $y = -54(b-a)(c-1)t^2((a-1)a(b-1)bc(c-a)(c-b)t^2 - 4(b-1)bc(c-a)t + 4)$
N'_{45}	$x = 18t((a-1)a(c-b)t - 2)((b-1)bc(c-a)t - 2)$ $y = -54(b-a)(c-1)t^2((a-1)a(b-1)bc(c-a)(c-b)t^2 - 4)$

28. FIBRATION 20

This fibration corresponds to the elliptic divisor $N_{23} + N_{12} + 2T_2 + 2N_2 + 2T_0 + 2N_4 + 2T_4 + 2N_{45} + 2T_5 + N_{35} + N'_{24}$, in the terminology of section 2. The translation by the section T_4 in that fibration takes it to the divisor $N_4 + N_3 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$, and we shall work with this elliptic divisor.

The corresponding elliptic fibration is obtained from fibration 2 by a 2-neighbor step. We may take T_4 to be the zero section. This fibration has a D_{10} and four A_1 fibers. The Mordell-Weil group has rank 1 and is generated by the section T_{13} of height 1. The resulting sublattice of $\text{NS}(X)$ has rank 17 and discriminant $4 \cdot 2^4 \cdot 1 = 64$, and must therefore be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{20} = \frac{x_2}{t_2^2} + 4(a-1)bc(c-b)t_2.$$

A Weierstrass equation for this fibration is

$$\begin{aligned}
 y^2 = & x^3 - 2x^2 \left(t^3 - 2(3b^2c^2 - 2abc^2 - 2bc^2 - 2ac^2 + 3c^2 - 2ab^2c - 2b^2c \right. \\
 & - 2a^2bc + 12abc - 2bc - 2a^2c - 2ac + 3a^2b^2 - 2ab^2 - 2a^2b - 2ab \\
 & + 3a^2) t^2 - 16(ab^2c^4 - b^2c^4 - a^2bc^4 + abc^4 + a^2b^3c^3 - 3ab^3c^3 + b^3c^3 \\
 & - a^3b^2c^3 + 3a^2b^2c^3 - ab^2c^3 + b^2c^3 - a^3bc^3 + 3a^2bc^3 - 3ab^2c^3 - a^3c^3 \\
 & + a^2c^3 - a^2b^4c^2 + ab^4c^2 + a^3b^3c^2 - a^2b^3c^2 + 3ab^3c^2 - b^3c^2 + 3a^3b^2c^2 \\
 & - 10a^2b^2c^2 + 3ab^2c^2 - a^4bc^2 + 3a^3bc^2 - a^2bc^2 + abc^2 + a^3c^2 - a^2c^2 \\
 & + a^2b^4c - ab^4c - 3a^3b^3c + 3a^2b^3c - ab^3c + a^4b^2c - a^3b^2c + 3a^2b^2c \\
 & - ab^2c + a^4bc - 3a^3bc + a^2bc + a^3b^3 - a^2b^3 - a^4b^2 + a^3b^2) t \\
 & - 32(a-1)ab(b-a)(c-1)c(c-b)(abc^2 + b^2c + a^2bc + bc + a^2c \\
 & - 2ac + ab^2 - 2a^2b + ab - 2bc^2 + ac^2 - 2ab^2c) \Big) \\
 & + x(t - 4a(b-a)(c-1))(t + 4(a-1)ab(c-b))(t + 4(a-1)c(c-b)) \cdot \\
 & (t - 4b(b-a)(c-1)c) \left(t^2 - 4t(3b^2c^2 - abc^2 - 4bc^2 - ac^2 + 3c^2 - 4ab^2c \right. \\
 & - b^2c - a^2bc + 12abc - bc - a^2c - 4ac + 3a^2b^2 - ab^2 - 4a^2b - ab + 3a^2) \\
 & - 16(b^2c^2 - ac^2 + ab^2c - b^2c - a^2bc + ac + a^2b - ab) \cdot \\
 & (abc^2 - bc^2 - ab^2c + bc + a^2c - ac + ab^2 - a^2b) \Big) \\
 & + 4(b-1)^2(c-a)^2(t - 4a(b-a)(c-1))^2(t + 4(a-1)ab(c-b))^2 \cdot \\
 & (t + 4(a-1)c(c-b))^2(t - 4b(b-a)(c-1)c)^2
 \end{aligned}$$

The reducible fibers are as follows:

Position	Reducible fiber	type
$t = \infty$	$N_4 + N_3 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1$ $+ 2N_{15} + 2T_5 + N_{35} + N'_{24}$	D_{10}
$t = -4ab(a-1)(c-b)$	$N_{34} + N_{34}^{(20)}$	A_1
$t = 4bc(b-a)(c-1)$	$N''_{13} + N_{13}^{(20)}$	A_1
$t = 4a(b-a)(c-1)$	$N_{14}^{(20)} + N''_{14}$	A_1
$t = -4c(a-1)(c-b)$	$N'_{25} + N_{25}^{(20)}$	A_1

The equation for the section T_{13} , which is a generator of the Mordell-Weil group, is given by:

$$\begin{aligned}
 x &= 0 \\
 y &= -2(b-1)(c-a)(t - 4a(b-a)(c-1))(t + 4(a-1)ab(c-b)) \cdot \\
 & (t + 4(a-1)c(c-b))(t - 4b(b-a)(c-1)c)
 \end{aligned}$$

29. FIBRATION 20A

This fibration corresponds to the elliptic divisor $N'_{24} + N_{35} + 2T_{14} + 2N_{14} + 2T_1 + 2N_{12} + 2T_2 + 2N_2 + 2T_0 + N_3 + N_5$. It may be obtained from fibration 11 by a 2-neighbor step. However, translation by the section $-T_{14}$ in fibration 11 transforms this divisor into $N'_{24} + N_{35} + 2T_5 + 2N_5 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$. This is related to the (modified) elliptic divisor $N_4 + N_3 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$ of fibration 20 by an element of $\text{Aut}(D')$, so this fibration is not new.

30. FIBRATION 20B

This fibration corresponds to the elliptic divisor $N_{12} + N_{23} + 2T_2 + 2N_2 + 2T_0 + 2N_3 + 2T_{13} + 2N_{45} + 2T_5 + N_{35} + N'_{24}$. It is a 2-neighbor step away from fibration 2. But translation by the section $-T_{13}$ turns this

divisor into the divisor $N_3 + N_4 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + 2N_{15} + 2T_5 + N_{35} + N'_{24}$, which equals the (modified) elliptic divisor of fibration 20. Hence this fibration is not new.

31. FIBRATION 21

This fibration corresponds to the elliptic divisor $T_5 + N_{15} + T_1 + N_{12} + T_2 + N_2 + T_0 + N_3 + T_{13} + N_{45}$. It is obtained from fibration 2 by a 2-neighbor step.

We may take N'_{24} as the zero section. This elliptic fibration has an A_9 fiber, three A_1 fibers, a 2-torsion section and Mordell-Weil rank 3. The divisors N_4 , N_{14} and N_{23} have the intersection pairing

$$\frac{1}{5} \begin{pmatrix} 8 & 6 & 2 \\ 6 & 12 & 4 \\ 2 & 4 & 8 \end{pmatrix}.$$

The determinant of the height pairing matrix is $16/5$. This implies that the sublattice of $\text{NS}(X)$ generated by these sections, along with the torsion section and the trivial lattice, has rank 17 and discriminant $10 \cdot 2^3 \cdot (16/5)/2^2 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{21} = \frac{y_2 - 8a(b-1)(c-1)(b-a)(c-a)t_2^2(t_2-1)}{2t_2(x_2 - 4a(b-a)(c-1)t_2^2)}.$$

A Weierstrass equation for this fibration is

$$\begin{aligned} y^2 = & x^3 + x^2 \left(t^4 + 2(2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - 3abc - bc \right. \\ & + 2a^2c + 2ac - ab^2 + 2a^2b + 2ab - 3a^2)t^2 + 8a(b-1)(b-a)(c-1)(c-a)t \\ & + (b^2c^4 - 2abc^4 + a^2c^4 - 2ab^3c^3 + 4a^2b^2c^3 + 2ab^2c^3 - 2b^2c^3 - 2a^3bc^3 \\ & - 2a^2bc^3 + 2abc^3 + a^2b^4c^2 - 2a^3b^3c^2 + 2a^2b^3c^2 + 4ab^3c^2 + a^4b^2c^2 \\ & - 2a^3b^2c^2 - 15a^2b^2c^2 - 2ab^2c^2 + b^2c^2 + 12a^3bc^2 + 6a^2bc^2 - 6a^3c^2 \\ & - 2a^2b^4c + 2a^3b^3c - 2a^2b^3c - 2ab^3c + 6a^3b^2c + 12a^2b^2c - 6a^4bc \\ & \left. - 10a^3bc - 6a^2bc + 4a^4c + 4a^3c + a^2b^4 - 6a^3b^2 + 4a^4b + 4a^3b - 3a^4) \right) \\ & + 16abc(a-1)(b-1)(c-1)(b-a)(c-a)(c-b)(t+ab-a)(t+c-a) \cdot \\ & (t-bc+ac+b-a)x \end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_5 + N_{15} + T_1 + N_{12} + T_2 + N_2$ $+ T_0 + N_3 + T_{13} + N_{45}$	A_9
$t = a - c$	$N_{25}^{(21)} + N'_{25}$	A_1
$t = -a(b-1)$	$N_{34}^{(21)} + N_{34}$	A_1
$t = (b-a)(c-1)$	$T_{14} + N''_{13}$	A_1

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N'_{24}	$x = y = \infty$
$V^{(21)}$	$x = y = 0$
N_4	$x = 4(a-1)(b-1)b(b-a)c(c-a)(c-b)$ $y = -4(a-1)(b-1)b(b-a)c(c-a)(c-b)(t^2 + 2a(c-1)t - (bc^2 - ac^2 + ab^2c - 2b^2c - a^2bc - abc + bc + 2a^2c + ab^2 - a^2))$
N_{14}	$x = -4a(b-1)(c-1)(c-a)(t+b-a)^2$ $y = -4a(b-1)(c-1)(c-a)(t+b-a)(t^3 + (b-a)t^2 - (bc^2 - ac^2 + ab^2c - 2b^2c - a^2bc + abc + bc + ab^2 - 2ab + a^2)t - (b-a) \cdot (2abc^2 - bc^2 - ac^2 - ab^2c - a^2bc + abc + bc + ab^2 - 2ab + a^2))$
N_{23}	$x = 4(a-1)a(b-1)b(c-1)c(c-a)(c-b)$ $y = 4(a-1)a(b-1)b(c-1)c(c-a)(c-b)(t^2 + 2(b-a)t + 2abc^2 - bc^2 - ac^2 - ab^2c - a^2bc + abc + bc + ab^2 - 2ab + a^2)$

The class of the torsion section $V^{(21)}$ is $H - N_0 - N_1 - N_{24} - N_{35} + T_{25}$: this is obtained by computing its intersections with a basis of $NS(X) \otimes \mathbb{Q}$.

32. FIBRATION 22

This fibration corresponds to the elliptic divisor $T_5 + N_{45} + T_4 + N_4 + T_0 + N_2 + T_2 + N_{12} + T_1 + N_{15}$. It is obtained from fibration 2 by a 2-neighbor step.

We may take N'_{24} to be the zero section. The elliptic fibration has an A_9 fiber, an A_1 fiber, and Mordell-Weil rank 5. The sections $N_3, N_{13}, N_{23}, N_{34}$ and T_{13} have the intersection pairing below.

$$\frac{1}{5} \begin{pmatrix} 8 & 6 & 2 & 4 & 2 \\ 6 & 12 & 4 & 8 & 4 \\ 2 & 4 & 8 & 6 & 8 \\ 4 & 8 & 6 & 12 & 6 \\ 2 & 4 & 8 & 6 & 13 \end{pmatrix}.$$

The determinant of the height pairing matrix is $16/5$. Therefore the sublattice of $NS(X)$ generated by these sections, along with the trivial lattice, has rank 17 and discriminant $10 \cdot 2 \cdot (16/5) = 64$. Therefore it must be all of $NS(X)$.

An elliptic parameter is given by

$$w = \frac{y_2}{t_2 x_2}.$$

A Weierstrass equation for this elliptic fibration is given by

$$\begin{aligned}
y^2 = & x^3 + x^2 \left(t^4 + 8(2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - bc - a^2c + 2ac \right. \\
& - ab^2 + 2a^2b - ab)t^2 + 16(b^2c^4 - 2abc^4 + a^2c^4 - 2ab^3c^3 + 4ab^2c^3 - 2b^2c^3 \\
& + 2a^3bc^3 - 2a^2bc^3 + 2abc^3 - 2a^3c^3 + a^2b^4c^2 - 2a^3b^3c^2 + 4a^2b^3c^2 \\
& + a^4b^2c^2 - 2a^3b^2c^2 - 4a^2b^2c^2 - 2ab^2c^2 + b^2c^2 - 2a^4bc^2 + 4a^3bc^2 + a^4c^2 \\
& - 2a^2b^4c + 2a^3b^3c - 2a^2b^3c + 2ab^3c + 4a^2b^2c - 2ab^2c - 2a^3bc + a^2b^4 \\
& \left. - 2a^2b^3 + a^2b^2) \right) - 1024(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b) \cdot \\
& (bc + ac - c - ab + b - a)t^2x \\
& + 65536(a-1)^2a^2(b-1)^2b^2(b-a)^2(c-1)^2c^2(c-a)^2(c-b)^2t^2.
\end{aligned}$$

This has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$T_5 + N_{45} + T_4 + N_4 + T_0 + N_2 + T_2 + N_{12} + T_1 + N_{15}$	A_9
$t = 0$	$T_3 + N''_{14}$	A_1

Generators for the Mordell-Weil group are in the table below.

Section	Equation
N'_{24}	$x = y = \infty$
N_3	$x = 64(a-1)(b-1)b(b-a)c(c-a)(c-b)$ $y = -64(a-1)(b-1)b(b-a)c(c-a)(c-b)(t^2 - 4(bc^2 - ac^2 + ab^2c - 2b^2c - a^2bc + bc + a^2c + ab^2 - ab))$
N_{13}	$x = -16(a-1)a(c-1)(c-b)(t-2bc+2ac)(t+2bc-2ac)$ $y = -16(a-1)a(c-1)(c-b)(t^4 - 4t^2(b^2c^2 - 4ab^2c^2 + bc^2 + 3a^2c^2 - ac^2 + ab^2c - 2b^2c - a^2bc + 2abc + bc - a^2c + ab^2 - ab) - 16(b-a)^2c^2(2ab^2c^2 - bc^2 - 2a^2c^2 + ac^2 - ab^2c + a^2bc - 2abc + bc + a^2c + ab^2 - ab))$
N_{23}	$x = 64(a-1)a(b-1)b(c-1)c(c-a)(c-b)$ $y = 64(a-1)a(b-1)b(c-1)c(c-a)(c-b)(t^2 + 4(2ab^2c^2 - bc^2 - ac^2 - ab^2c - a^2bc + bc + a^2c + ab^2 - ab))$
N_{34}	$x = -16(b-1)(b-a)c(t-2ac+2ab)(t+2ac-2ab)$ $y = 16(b-1)(b-a)c(t^4 + 4t^2(2ab^2c^2 - bc^2 - a^2c^2 - ac^2 - ab^2c + a^2bc + 2abc + bc - a^2c - a^2b^2 - ab^2 + 2a^2b - ab) - 16a^2(c-b)^2(bc^2 - ac^2 - ab^2c + a^2bc + 2abc - bc - a^2c - ab^2 + ab))$
Q	$x = 0, \quad y = 256abc(a-1)(b-1)(c-1)(b-a)(c-a)(c-b)t$

Note that we replaced the section T_{13} by a section Q for which $x_Q = 0$, since this section has a much simpler expression. It is easily checked that $T_{13} = Q + N_{23}$ in the Mordell-Weil group, so the five non-zero sections above do form a basis of the Mordell-Weil group.

33. FIBRATION 23

This fibration corresponds to the elliptic divisor $N_{12} + N_{24} + 2T_2 + 2N_2 + 2T_0 + 2N_3 + 2T_3 + 2N_{35} + T_{14} + T''_{14}$, in the terminology of fibration 3. We may translate by the torsion section T_3 in that fibration to get a new elliptic divisor $N_3 + N_5 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + 2N_{14} + T_{14} + T''_{14}$, which we shall work with. The corresponding elliptic fibration is obtained from fibration 3 by a 2-neighbor step.

We may take T_3 to be the zero section. This fibration has a D_9 fiber, six A_1 fibers, a 2-torsion section T_5 and Mordell-Weil rank 0. The trivial lattice and the 2-torsion section span a sublattice of $\text{NS}(X)$ of rank 17 and discriminant $4 \cdot 2^6/2^2 = 64$, which must therefore be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{23} = \frac{x_3}{4t_3^2} - \frac{act_3}{4} + \frac{4(a-1)b(b-a)(c-1)(c-b)}{t_3} - \frac{1}{3} \left(2ab^2c^2 - bc^2 - 3a^2c^2 + 2ac^2 - ab^2c + 2b^2c + 2a^2bc - 6abc + 2bc + 2a^2c - ac + 2ab^2 - 3b^2 - a^2b + 2ab \right).$$

This fibration was studied in [Ku]. The resulting Weierstrass equation is independent of the level 2 structure, and can be expressed as

$$y^2 = x^3 - 2 \left(t^3 - \frac{I_4}{12}t + \frac{I_2 I_4 - 3I_6}{108} \right) x^2 + \left(\left(t^3 - \frac{I_4}{12}t + \frac{I_2 I_4 - 3I_6}{108} \right)^2 + I_{10} \left(t - \frac{I_2}{24} \right) \right) x.$$

Here I_2, I_4, I_6, I_{10} are the Igusa-Clebsch invariants of the genus 2 curve C , described in Section 3.5.

The reducible fibers are described in the following table.

Position	Reducible fiber	type
$t = \infty$	$N_3 + N_5 + 2T_0 + 2N_2 + 2T_2 + 2N_{12}$ $+ 2T_1 + 2N_{14} + T_{14} + T_{14}'''$	D_9
$t = (abc^2 - 2bc^2 + ac^2 + ab^2c$ $+ b^2c - 2a^2bc + bc + a^2c$ $- 2ac - 2ab^2 + a^2b + ab)/3$	$N_{25}' + N_{25}^{(23)}$	A_1
$t = (-2abc^2 + bc^2 + ac^2 + ab^2c$ $- 2b^2c + a^2bc + bc + a^2c$ $- 2ac + ab^2 - 2a^2b + ab)/3$	$N_{23}^{(23)} + N_{23}'$	A_1
$t = (-2abc^2 + bc^2 + ac^2 + ab^2c$ $+ b^2c + a^2bc - 2bc - 2a^2c$ $+ ac - 2ab^2 + a^2b + ab)/3$	$N_{15}''' + N_{15}^{(23)}$	A_1
$t = (abc^2 + bc^2 - 2ac^2 - 2ab^2c$ $+ b^2c + a^2bc - 2bc + a^2c$ $+ ac + ab^2 - 2a^2b + ab)/3$	$N_{34} + N_{34}^{(23)}$	A_1
$t = (abc^2 + bc^2 - 2ac^2 + ab^2c$ $- 2b^2c - 2a^2bc + bc + a^2c$ $+ ac + ab^2 + a^2b - 2ab)/3$	$N_{13}^{(23)} + N_{13}'''$	A_1
$t = (abc^2 - 2bc^2 + ac^2 - 2ab^2c$ $+ b^2c + a^2bc + bc - 2a^2c$ $+ ac + ab^2 + a^2b - 2ab)/3$	$N_{45}^{(23)} + N_{45}$	A_1

The torsion section T_5 is given by $x = y = 0$.

34. FIBRATION 24

This fibration corresponds to the elliptic divisor $2T_{15} + 4N_5 + 6T_0 + 3N_3 + 5N_2 + 4T_2 + 3N_{12} + 2T_1 + N_{14}$. Translation by the section $-T_{15}$ of fibration 11 transforms this to the elliptic divisor $2T_5 + 4N_5 + 6T_0 + 3N_3 + 5N_2 + 4T_2 + 3N_{12} + 2T_1 + N_{14}$, which we shall work with. It is obtained from fibration 11 by a 2-neighbor step.

We may take T_4 to be the zero section. This fibration has an E_8 fiber, six A_1 fibers, and Mordell-Weil rank 1. The section T_{14} has height $4 - 6(1/2) = 1$, and so the sublattice of $\text{NS}(X)$ it generates with the trivial lattice has rank 17 and discriminant $1 \cdot 2^6 \cdot 1 = 64$. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{24} = \frac{(a-1)(c-b)x_{11}}{abc(b-1)(c-1)(b-a)(c-a)} + 4t_{11} - 2(bc - ac - c - ab + b + a).$$

A Weierstrass equation for this elliptic fibration is

$$\begin{aligned}
y^2 = & x^3 - 48 I_4 x^2 - 12 x (3 t^2 - 2 I_2) (3 t^2 I_4 + 432 t I_5 - 48 I_6 + 14 I_2 I_4) \\
& + 16 \left(729 I_5 t^7 - 27 (3 I_6 - I_2 I_4) t^6 - 1458 I_2 I_5 t^5 + 54 (3 I_2 I_6 + 6 I_4^2 - I_2^2 I_4) t^4 \right. \\
& + 972 (32 I_4 + I_2^2) I_5 t^3 - 36 (96 I_4 I_6 + 3 I_2^2 I_6 - 20736 I_5^2 - 20 I_2 I_4^2 - I_2^3 I_4) t^2 \\
& \left. - 216 I_5 (768 I_6 - 160 I_2 I_4 + I_2^3) t + 8 (1152 I_6^2 - 480 I_2 I_4 I_6 + 3 I_2^3 I_6 + 50 I_2^2 I_4^2 - I_2^4 I_4) \right)
\end{aligned}$$

It has the following reducible fibers.

Position	Reducible fiber	type
$t = \infty$	$2T_{15} + 4N_5 + 3N_3 + 6T_0 + 5N_2 + 4T_2$ $+ 3N_{12} + 2T_1 + N_{14}$	E_8
$t = -2(b c - a c - c - a b + b + a)$	$N_{35}^{(11)} + N_{35}^{(24)}$	A_1
$t = -2(b c + a c - c - a b - b + a)$	$N_{24}^{(11)} + N_{24}^{(24)}$	A_1
$t = 2(b c - a c - c + a b - b + a)$	$N_{34} + N_{34}^{(24)}$	A_1
$t = 2(b c - a c + c - a b - b + a)$	$N_{25}' + N_{25}^{(24)}$	A_1
$t = 2(b c + a c - c - a b + b - a)$	$N_{45}^{(24)} + N_{45}^{(11)}$	A_1
$t = -2(b c - a c + c + a b - b - a)$	$N_{23}^{(24)} + N_{23}^{(11)}$	A_1

The section T_{14} is given by

$$\begin{aligned}
x &= (3 t^2 - 2 I_2)^2 / 4 \\
y &= -(27 t^6 - 54 I_2 t^4 + (-576 I_4 + 36 I_2^2) t^2 - 27648 I_5 t + 3072 I_6 - 640 I_2 I_4 - 8 I_2^3) / 8.
\end{aligned}$$

As it stands, the Weierstrass equation above is almost invariant of the level 2 structure. The only problem is the appearance of the square root I_5 of I_{10} rather than the genuine covariant I_{10} . This can easily be remedied, as follows. Letting

$$\alpha = \frac{I_2^3 \cdot I_4}{I_{10}}, \quad \beta = \frac{I_2^2 \cdot I_6}{I_{10}}, \quad \gamma = \frac{I_2^5}{I_{10}}$$

be a complete set of invariants of the genus 2 curve, and $\mu = I_5 / I_2^2$, we can scale t by μ and x and y by μ^4 and μ^6 respectively, to make the convert the above equation into the Weierstrass equation below, which depends solely on α, β, γ .

$$\begin{aligned}
y^2 = & x^3 - 48 \alpha \gamma x^2 - 12 x \gamma (3 t^2 - 2 \gamma) (3 \alpha t^2 + 432 \gamma t + (-48 \beta + 14 \alpha) \gamma) \\
& + 16 \left(729 \gamma^2 t^7 - 27 (3 \beta - \alpha) \gamma^2 t^6 - 1458 \gamma^3 t^5 + 54 \gamma^2 (3 \beta \gamma - \alpha \gamma + 6 \alpha^2) t^4 \right. \\
& + 972 \gamma^3 (\gamma + 32 \alpha) t^3 - 36 \gamma^3 (3 \beta \gamma - \alpha \gamma - 20736 \gamma + 96 \alpha \beta - 20 \alpha^2) t^2 \\
& \left. - 216 \gamma^4 (\gamma + 768 \beta - 160 \alpha) t + 8 \gamma^4 (3 \beta \gamma - \alpha \gamma + 1152 \beta^2 - 480 \alpha \beta + 50 \alpha^2) \right)
\end{aligned}$$

35. FIBRATION 24A

This elliptic fibration corresponds to the divisor

$$(-4, -5, 0, 0, 0, -5, 0, -3, -1, 0, 0, -2, -4, -1, -1, 0, 7),$$

and is therefore obtained by a 2-neighbor step from the original elliptic divisor of fibration 11A. However, we used the translation by $-T_{15}$ in fibration 2 to obtain the new elliptic divisor for fibration 11A. Therefore, we apply this automorphism of $NS(X)$, which comes from an automorphism of X , to our elliptic divisor and obtain the elliptic divisor

$$(-10, -3, 0, 0, 0, -5, 0, -1, -3, 0, 0, -2, -2, -3, -1, 0, 9).$$

Now there are two explicit automorphisms in $\text{Aut}(D')$ which takes the new elliptic divisor for the elliptic divisor of fibration 11A to that of fibration 11. One of them is the permutation of Weierstrass points, given by $(0, 1, 5)(2, 4)$ in cycle notation. This transformation converts our elliptic divisor to the divisor class

$$(-10, -5, 0, -1, -2, 0, 0, 0, 0, -3, -3, -2, -3, 0, -1, 0, 9).$$

This elliptic divisor is related to that of fibration 11 by a 2-neighbor step. Finally, in the elliptic fibration 11, we may translate by the section T_{14} . This takes the elliptic divisor above to

$$(-7, -4, 0, 0, -3, 0, 0, -1, 0, -2, -2, -2, -3, 0, -1, -1, 7),$$

which is the (modified) elliptic divisor of fibration 24. Therefore this fibration is not new.

36. FIBRATION 24B

This elliptic fibration corresponds to the elliptic divisor $2T_1 + 4N_{12} + 6T_2 + 3N_{23} + 5N_2 + 4T_0 + 3N_3 + 2T_{13} + N_{45}$. It may be obtained by a 2-neighbor step from fibration 16. However, the translation by $-T_{13}$ in that fibration sends this to the divisor $2T_0 + 4N_2 + 6T_2 + 3N_{23} + 5N_{12} + 4T_1 + 3N_{15} + 2T_5 + N_{45}$, which is in the $\text{Aut}(D')$ orbit of the (modified) elliptic divisor $2T_5 + 4N_5 + 6T_0 + 3N_3 + 5N_2 + 4T_2 + 3N_{12} + 2T_1 + N_{14}$ of fibration 24. Therefore this fibration is not new.

37. FIBRATION 24C

The elliptic divisor $[-3, -2, -4, 0, -3, -1, 0, -1, 0, -1, -7, 0, 0, 0, -2, -2, 7]$ corresponding to this elliptic fibration is in the $\text{Aut}(D')$ -orbit of the (modified) elliptic divisor $2T_5 + 4N_5 + 6T_0 + 3N_3 + 5N_2 + 4T_2 + 3N_{12} + 2T_1 + N_{14}$ of fibration 24. Hence this fibration is not new.

38. FIBRATION 25

This fibration corresponds to the elliptic divisor $T_5 + T_{15} + 2N_5 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$. It is obtained from fibration 11 by a 2-neighbor step. This is the most complicated of the twenty-five elliptic fibrations, in terms of the Weierstrass equation, the elliptic parameter and the sections, and it is the one of highest Mordell-Weil rank.

We may take N_{35}^{11} to be the zero section. This fibration has a D_9 fiber and Mordell-Weil rank 6. The divisors N_{45} , N_{24}' , N_{34} , N_{23}' , T_{14} and N_{25}^{11} have the intersection pairing

$$\frac{1}{4} \begin{pmatrix} 12 & 4 & 8 & 4 & 6 & 4 \\ 4 & 12 & 8 & 4 & 2 & 4 \\ 8 & 8 & 16 & 8 & 8 & 8 \\ 4 & 4 & 8 & 12 & 6 & 4 \\ 6 & 2 & 8 & 6 & 7 & 2 \\ 4 & 4 & 8 & 4 & 2 & 12 \end{pmatrix}.$$

The determinant of the height pairing matrix is 16. This implies that the sublattice of $\text{NS}(X)$ generated by these sections along with the identity section and the components of the reducible fiber has rank 17 and discriminant 64. Therefore it must be all of $\text{NS}(X)$.

An elliptic parameter is given by

$$t_{25} = \left(\frac{3(y_{11} + 2(t_{11} + ac - a)(t_{11} - bc + ac)x_{11})}{x_{11} - 4t_{11}(t_{11} + ac - b)(t_{11} - bc + ac + ab - a)(t_{11} - bc + ac + c - a)} \right. \\ \left. - 6(a - 1)(c - b)t_{11} + 2(2abc^2 - bc^2 - 3a^2c^2 + 2ac^2 - ab^2c + 2b^2c \right. \\ \left. + 2a^2bc - 3abc - bc + 2a^2c - ac - ab^2 - a^2b + 2ab) \right).$$

A Weierstrass equation for this fibration is given by

$$y^2 = x^3 - 6x^2 \left(t^3 - 3I_4 t + (6I_6 - 2I_2 I_4) \right) + 69984x I_{10} (8t - I_2) \\ + 419904 I_{10} \left(t^4 + I_2 t^3 + 6I_4 t^2 + (-48I_6 + 13I_2 I_4) t + 6I_2 I_6 + 9I_4^2 - 2I_2^2 I_4 \right)$$

where I_2, I_4, I_6, I_{10} are the Igusa-Clebsch invariants of the curve C , described in section 3.5.

The D_9 fiber $T_{15} + T_5 + 2N_5 + 2T_0 + 2N_2 + 2T_2 + 2N_{12} + 2T_1 + N_{13} + N_{14}$ is at $t = \infty$, with T_{15} being the identity component.

Next, we write down six sections which generate the Mordell-Weil lattice. In fact, we will not use the sections $P_1 = N_{45}$, $P_2 = N'_{24}$, $P_3 = N_{34}$, $P_4 = N'_{23}$, $P_5 = T_{14}$ and $P_6 = N_{25}^{11}$, but the linear combinations of them given by

$$\begin{aligned} Q_1 &= -P_3 + P_5 + P_6 \\ Q_2 &= P_4 - P_5 \\ Q_3 &= P_5 \\ Q_4 &= -P_1 + P_5 \\ Q_5 &= P_2 - P_3 + P_5 \\ Q_6 &= -P_3 + P_5 \end{aligned}$$

where the addition refers to addition of points on the elliptic curve over $k(t)$ and not addition in $\text{NS}(X)$: to get the latter, one has to add some linear combinations of fibral components which may be easily deduced.

The advantage of these linear combinations is that these sections now all have height $7/4$, which is the smallest norm in the Mordell-Weil lattice (in fact, up to sign they are all the minimal vectors of the Mordell-Weil lattice), while still generating it.

We now list just the x -coordinates of these sections, for brevity.

Section	Formula for $x(t)$
Q_1	$-648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc+ac-c-ab-b+a)$
Q_2	$-648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc-ac+c+ab-b-a)$
Q_3	$-648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc-ac-c-ab+b+a)$
Q_4	$648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc+ac-c-ab+b-a)$
Q_5	$648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc-ac+c-ab-b+a)$
Q_6	$648(a-1)a(b-1)b(b-a)(c-1)c(c-a)(c-b)(bc-ac-c+ab-b+a)$

39. APPENDIX A

In this appendix we describe some basic computations which are needed in the main body of the paper, to convert between different elliptic fibrations. We learned the technique of “elliptic hopping” using 2-neighbors from Noam Elkies.

39.1. Global sections of some divisors on an elliptic curve. We describe, for completeness, the global sections of some simple line bundles on an elliptic curve E . In this paper, we use these results in the following manner: we have a K3 surface X over a field K with an elliptic fibration over \mathbb{P}_K^1 , with a zero section O , and perhaps some other sections P, Q etc. Let F be the class of a fiber. Then for an effective divisor $D = mO + nP + kQ + G$, where G is the class of an effective vertical divisor (i.e. fibers and components of reducible fibers), we would like to compute the global sections of $\mathcal{O}_X(D)$. Any such global section gives a section of the generic fiber, which is an elliptic curve E over $K(t)$. Therefore, if we have a basis $\{s_1, \dots, s_r\}$ of the global sections of $mO + nP + kQ$ over $K(t)$, we can assume that any global section of $\mathcal{O}_X(D)$ must be of the form

$$b_1(t)s_1 + \dots + b_r(t)s_r$$

with $b_i(t) \in K(t)$. We can then use the information from G , which gives us conditions about the zeros and poles of the functions a_i , to find the linear space cut out by $H^0(X, \mathcal{O}_X(D))$.

We also describe what happens when we make the transformation to the new elliptic parameter, and how to convert the resulting genus 1 curve to a double cover of \mathbb{P}^1 branched at 4 points. Later we will describe how to convert these to Weierstrass form.

First, we consider the case $D = 2O$. Then 1 and x are a basis for the global sections. Therefore, for an elliptic K3 surface X given by a minimal proper model of

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

with $a_i \in K[t]$ of degree at most $2i$, and an elliptic divisor $F' = 2O + G$ with G effective and vertical, we obtain two global sections 1 and $a(t) + b(t)x$ for some fixed $a(t), b(t)$. The ratio of these two sections gives the new elliptic parameter u . Therefore we set $x = (u - b(t))/a(t)$ and substitute into the Weierstrass equation, to obtain

$$y^2 = g(t, u)$$

Since the generic fiber of this surface over \mathbb{P}_u^1 is a curve of genus 1 (by construction of F'), we must have after absorbing square factors into y^2 , that g is a polynomial of degree 3 or 4 in t .

Next, consider $D = O + P$ where $P = (x_0, y_0)$ is not a 2-torsion section. Then 1 and $(y + y_0)/(x - x_0)$ are global sections of $\mathcal{O}_E(D)$. For an elliptic K3 surface as above and $F' = O + P + G$, we obtain global sections 1 and $a(t) + b(t)(y + y_0)/(x - x_0)$. Setting the latter equal to u as before, we solve for y to get

$$y = \frac{(u - a(t))(x - x_0)}{b(t)} - y_0$$

Substituting into the Weierstrass equation we get

$$\left(\frac{(u - a(t))(x - x_0)}{b(t)} - y_0 \right)^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

Note that when $x = x_0$, the left and right hand sides of this equation both evaluate to y_0^2 . Therefore their difference is a multiple of $(x - x_0)$ in the polynomial ring $K(t)[x]$, which may be cancelled to give an equation $g(x, t, u) = 0$ which is quadratic in x . Completing the square converts it to an equation

$$x^2 = h(t, u)$$

and, as before, we can argue that after absorbing square factors into the left hand side, h must be cubic or quartic in t .

The final case we need to consider is $D = O + T$, where T is a 2-torsion section, which we may take to be $(0, 0)$, while taking an equation of X of the form

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x.$$

Now 1 and y/x are sections of $\mathcal{O}_E(D)$. For an elliptic K3 surface and $F' = O + T + G$, we obtain global sections 1 and $a(t) + b(t)y/x$. Setting the latter equal to u , we obtain

$$y = (u - a(t))x/b(t).$$

Substituting into the Weierstrass equation results in

$$\left(\frac{(u - a(t))x}{b(t)} \right)^2 = x^3 + a_2(t)x^2 + a_4(t)x$$

We cancel a factor of x from both sides, obtaining a quadratic equation, and proceed as in the previous case.

In fact, these are all the cases we need to consider. For if we have an elliptic divisor F' on an elliptic K3 surface with fiber class F satisfying $F' \cdot F = 2$, then decomposing F' into horizontal and vertical components $F' = F'_h + F'_v$, we see that $F'_h \cdot F + F'_v \cdot F = F' \cdot F = 2$. But F'_v consists of components of fibers, by definition, so its intersection with F is zero. Therefore $F'_h \cdot F = 2$. The only possibilities are $2P$ and $P + Q$ for sections P, Q . We can translate the first one using an automorphism to get $2O$, and the second to get $O + T$ or $O + P$ depending on whether the class of the section $[P - Q]$ is 2-torsion or not.

39.2. Conversion to Weierstrass form. We recall how to convert a genus 1 curve over a field K with affine model

$$y^2 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

to Weierstrass form, given a K -rational point on the curve (references for this method are [Conn, Ru]). First, if the given point is at ∞ , then $a_4 = \alpha^2$ is a square, and we may change coordinates to $\eta = y/x^2$ and $\xi = 1/x$ to get a point $(0, \alpha)$ on

$$\eta^2 = a_0\xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4$$

So we may assume that the K -rational point is an affine point, and by translating x by x_0 in the equation above, we may assume that $(0, \beta)$ is a rational point.

Set $x = u/v$ and $y = (Av^2 + Buv + Cu^2 + Du^3)/v^2$ in the equation. We get

$$(Av^2 + Buv + Cu^2 + Du^3)^2 = a_4u^4 + a_3u^3v + a_2u^2v^2 + a_1uv^3 + \beta^2v^4$$

We successively solve for the coefficients, so that the difference of the two sides is divisible by u^3 .

In other words, we set

$$A = \beta, \quad B = \frac{a_1}{2\beta}, \quad C = \frac{4a_2\beta^2 - a_1^2}{8\beta^3}$$

Finally, we divide the resulting equation by u^3 , and choose $D = -2\beta$ so that the coefficients of v^2 and u^3 add to zero. This leads us to a Weierstrass equation of the form

$$v^2 + b_1uv + b_3v = u^3 + b_2u^2 + b_4u$$

which we may then convert to Weierstrass form.

40. APPENDIX B

In this section we explicitly describe the convex polytope \mathcal{P} which is used in Section 3.2. The point $(\alpha_0, \alpha_1, \dots, \alpha_{45}) \in \mathbb{R}^{16}$ of \mathcal{P} corresponds to the divisor $D = H - \sum \alpha_\mu N_\mu$ of $NS(X)$. These divisors must satisfy the linear constraints $D \cdot r' \geq 0$, for 316 different $r' \in S \otimes \mathbb{Q}$. We may scale the r' so it lies in S , which is integral, and therefore we may assume the inequalities are of the form $\sum c_\mu \alpha_\mu + d \geq 0$, with c_μ and d integers with no common divisor greater than 1.

The polytope \mathcal{P} has 30124888 vertices, and it is beyond our current limits of computation to enumerate these directly, within a reasonable amount of time. Hence, we must make use of the symmetry group of the polytope. The software `SymPol` [RS] outputs a list of 2961 vertices \mathcal{V}_0 of \mathcal{P} , which are exhaustive modulo the action of the (affine) symmetry group of \mathcal{P} in \mathbb{R}^{16} . However, the affine symmetry group of \mathcal{P} is an index 2 subgroup of the full symmetry group of D' , because the switch $\sigma \in \text{Aut}(D')$ is not an affine symmetry of \mathcal{P} . Taking the full symmetry group into account, we obtain a list \mathcal{V} of 1492 vertices. This computation using `SymPol` requires only about 33 minutes of computation time on a 3 gigahertz processor, using 5.4 gigabytes of memory. We now describe an alternate method to just verify the correctness of this list of vertices, which is simpler and less memory intensive (note that, however, this method relies on having the putative list of vertices). We thank Henry Cohn for suggesting this approach.

First, we may dualize the problem, to obtain a polytope \mathcal{Q} which has only 316 vertices, and a symmetry group G of size $6! \cdot 2^5 = 23040$. We have a list of 1492 faces of this polytope, and we would like to assert that it is complete up to symmetry. Since the polytope is convex and hence connected, it is enough to verify that for each of these 1492 faces F , and each codimension 1 facet F' of F , that the face lying on the opposite side of F' is in the orbit of one of the 1492 faces under the group G . There are 13394 such opposite faces, and we check whether each of these lies in such an orbit.

The computer files for checking these calculations are available from the [arXiv.org](https://arxiv.org) e-print archive. To access the auxiliary files, download the source file for the paper. This will produce not only the \LaTeX file for this paper, but also the computer code. The file `README.txt` gives an overview of the various computer files involved in the verification.

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REFERENCES

- [B] R. Borcherds, *Automorphism groups of Lorentzian lattices*, J. Algebra **111** (1987), 133–153.
- [CF] J. W. S. Cassels, *Prolegomena to a middlebrow arithmetic of curves of genus 2*, London Mathematical Society Lecture Note Series, **320**. Cambridge University Press, Cambridge, 1996.
- [Cl] A. Clebsch, *Zur Theorie der binären algebraischen Formen*, Verlag von B. G. Teubner, Leipzig, 1872.
- [Conn] I. Connell, *Addendum to a paper of K. Harada and M.-L. Lang: "Some elliptic curves arising from the Leech lattice"*, J. Algebra **125** (1989) no. 2, 298–310.
- [Conw] J. H. Conway, *Three lectures on exceptional groups*, in *Finite Simple Groups*, pp. 215–247, Academic Press, New York, 1971.
- [D] P. Deligne, *Relèvement des surfaces K3 en caractéristique nulle*, in *Surfaces Algébriques*, Lecture Notes in Math. **868**, pp. 58–79, Springer, Berlin-New York, 1981.
- [Hud] R. W. H. T. Hudson, *Kummer's quartic surface*, Cambridge University Press, Cambridge, 1905.
- [Hut] J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bull. Amer. Math. Soc. **7** (1901), 211–217.
- [I] J. Igusa, *Arithmetic variety of moduli for genus two*, Ann. of Math. (2) **72** (1960), 612–649.
- [Ke1] J. H. Keum, *Two extremal elliptic fibrations on Jacobian Kummer surfaces*, Manuscripta Math. **91** (1996), no. 3, 369–377.
- [Ke2] J. H. Keum, *Automorphisms of Jacobian Kummer surfaces*, Compositio Math. **107** (1997), no. 3, 269–288.
- [KK] J. H. Keum and S. Kondo, *The automorphism groups of Kummer surfaces associated with the product of two elliptic curves*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1469–1487.
- [Kl] F. Klein, *Ueber Configurationen, welche der Kummer'schen Fläche zugleich eingeschrieben und umgeschrieben sind*, Math. Ann. **27** (1886), 106–142.
- [Ko] S. Kondo, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom. **7** (1998), no. 3, 589–609.
- [Ku] A. Kumar, *K3 surfaces associated with curves of genus two*, Int. Math. Res. Not. IMRN **2008**, no. 6, Art. ID rnm165, 26 pp.
- [KS] M. Kuwata and T. Shioda, *Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface*, Algebraic geometry in East Asia – Hanoi 2005, 177–215, Adv. Stud. Pure Math. **50**, Math. Soc. Japan, Tokyo, 2008.
- [M] J.-F. Mestre, *Construction de courbes de genre 2 à partir de leurs modules*, Effective methods in algebraic geometry (Castiglione, 1990), 313–334, Progr. Math. **94**, Birkhäuser Boston, Boston, MA, 1991.
- [N] K. Nishiyama, *The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups*, Japan. J. Math. (New Series) **22** (1996), no. 2, 293–347.
- [O] K. Oguiso, *On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves*, J. Math. Soc. Japan **41** (1989), no. 4, 651–680.
- [PSS] I. Piatetski-Shapiro and I. R. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Izv. **5** (1971), 547–587.
- [RS] T. Rehn and A. Schürmann, *C++ Tools for Exploiting Polyhedral Symmetries*, Lecture Notes in Computer Science Vol. 6327 (2010), 295–298.
- [Ru] D. Rusin, Mathematical Atlas entry on *Maple input file for reducing a curve $y^2 = \text{quartic}$ to normal form*, <http://www.math.niu.edu/~rusin/known-math/96/quartic.maple>.
- [Sh1] T. Shioda, *Classical Kummer surfaces and Mordell-Weil lattices*, Algebraic geometry, 213–221, Contemp. Math. **422**, Amer. Math. Soc., Providence, RI, 2007.
- [Sh2] T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240.
- [St] H. Sterk, *Finiteness results for algebraic K3 surfaces*, Math. Z. **189** (1985), no. 4, 507–513.
- [V] Vinberg, *Some arithmetical discrete groups in Lobachevskii spaces*, in Proc. Internat. Colloq. on Discrete Subgroups of Lie Groups and Applications to Moduli (Bombay 1973), pp. 323–348. Oxford University Press, Bombay, 1975.

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